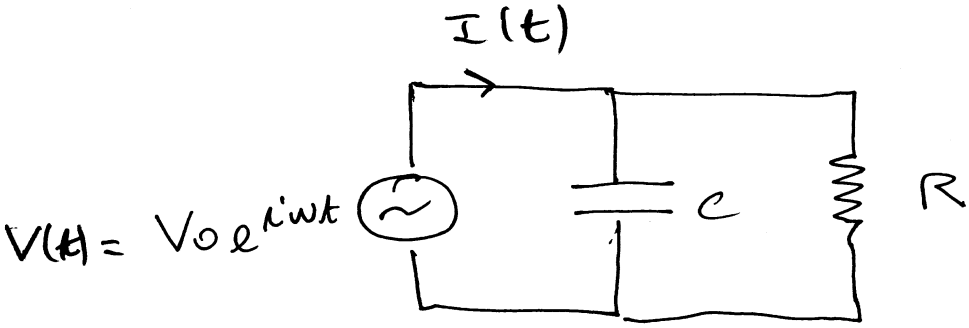
Parallel CR circuit



$$\frac{1}{Z_{total}} = \frac{1}{Z_{capacitor}} + \frac{1}{Z_{resistor}}$$

$$= \frac{1}{\left(\frac{1}{i\omega C}\right)} + \frac{1}{R}$$

$$= i\omega C + \frac{1}{R}.$$

$$\Rightarrow I = \frac{V}{Z_{total}}$$

$$= i\omega C V_0 e^{i\omega t} + \frac{V_0}{R} e^{i\omega t}$$

$$= i\omega C V_0 (\cos\omega t + i \sin\omega t)$$

$$+ \frac{V_0}{R} (\cos\omega t + i \sin\omega t)$$

Taking the real part.

$$I = -\omega C V_0 \sin\omega t + \frac{V_0}{R} \cos\omega t$$

current ~~into~~ capacitor

current through resistor

90° out of phase.

In complex notation

$$I(t) = i\omega C V_0 e^{i\omega t} + \frac{V_0}{R} e^{i\omega t}$$

$$= \omega C V_0 e^{i\frac{\pi}{2}} e^{i\omega t} + \frac{V_0}{R} e^{i\omega t}$$

$$= \omega C V_0 e^{i(\omega t + \frac{\pi}{2})} + \frac{V_0}{R} e^{i\omega t}$$

phase of current
into capacitor
is $\frac{\pi}{2}$ ahead of
that through
resistor.

If $\omega C << \frac{1}{R}$, current flows predominantly through the resistor; if $\omega C >> \frac{1}{R}$, it flows predominantly through the capacitor.

<u>Reason</u>: ω <u>large</u> $\Rightarrow$ capacitor only has time to partially charge before it discharges $\Rightarrow$ <u>easy</u> compared to passing current through R $\Rightarrow$ bulk of current flows through C.

For ω small $\Rightarrow$ capacitor has time to fully charge and then bulk of current flows through R.

<u>So</u>: if R represents a load, this is a way to pass low <u>frequency</u> components of a ~~...~~ voltage through the load.

Energy in AC circuits

- To build up a current I in an inductor, energy $U = \frac{1}{2} L I^2$ must be provided by the external circuit. This is stored as potential energy in the magnetic field of the inductor ⇒ when the current falls back to zero, this energy is returned to the circuit. So with an AC current, energy flows back and forth between the inductor and the rest of the circuit ⇒ average rate at which energy is taken from the circuit by the inductor is <u>zero</u>. So an inductor is a <u>non-dissipative</u> circuit element, no energy is "lost".

- Similarly, energy $U = \frac{1}{2}\frac{Q^2}{C}$

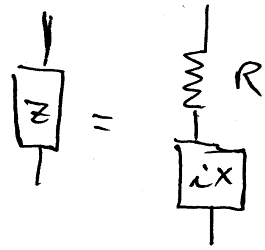
is required to charge a capacitor, it is stored as potential energy in the electric field in the capacitor, and it is recovered when the capacitor discharges. So energy flows in and out of the capacitor in an AC circuit, but no energy is lost $\Rightarrow$ a capacitor is a <u>non-dissipative</u> circuit element.

- But with a resistor, power $\frac{I^2}{R}$ $(=VI)$ is lost to create heat no matter which direction the current flows $\Rightarrow$ energy is <u>always</u> <u>dissipated</u> (mechanical analogue is friction).

- Inductors and capacitors have purely imaginary impedance ($Z_{\text{inductor}} = i\omega L$, $Z_{\text{capacitor}} = -\frac{i}{\omega C}$), a resistor has a real impedance.

- What about a circuit with an arbitrary impedance – how much power is "lost"?

- An arbitrary impedance can be decomposed as



$$\underbrace{Z = R}_{\text{real part, resistance}} + \underbrace{iX}_{\text{imaginary part, REACTANCE}}$$

We will show it is only the real part of Z that leads to energy loss – the REACTANCE does not cause any energy loss

Proof: if a generator producing a voltage ΔV is connected to the circuit

$$\Delta V = I Z = I (R + i X)$$

If $I_0 = I_0 e^{i \omega t}$

$$\Delta V = I_0 (\cos \omega t + i \sin \omega t)(R + i X)$$

BUT: the true current and potential difference is given by the real parts:

$$\therefore \quad I = I_0 \cos \omega t$$

$$\Delta V = I_0 R \cos \omega t - I_0 X \sin \omega t.$$

Then the power dissipated in the circuit is

$$P = \text{time average of } (I . \Delta V)$$

$$= \text{time average of } (I_0^2 R \cos^2 \omega t$$

$$- I_0^2 X \cos \omega t \sin \omega t)$$

Time average of $\cos^2 \omega t = \frac{1}{2}$ (it is always positive and varies between 0 + 1).

Time average of $\cos\omega t \cdot \sin\omega t = 0$

(it is sometimes positive, sometimes negative, cancels to zero over an period of oscillation).

So power consumed by circuit

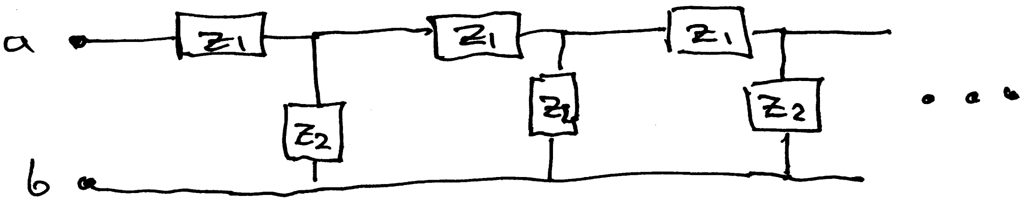
$$P = \frac{1}{2} I_0^2 R,$$

independent of the reactance X in $Z = R + iX$.

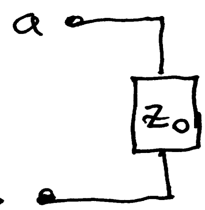
<u>So</u>: reactance does not produce any power loss in a circuit.

LADDER NETWORK

Consider an infinitely long ladder:

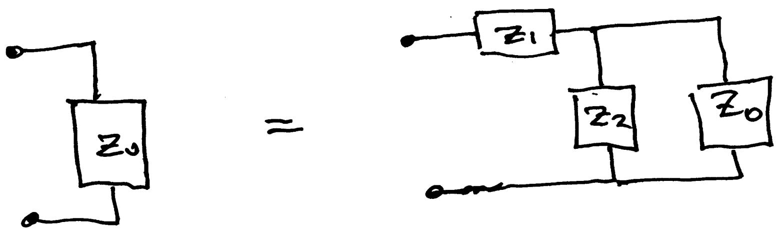


Denote the total impedance by Z_0:



But since the ladder is infinite, adding another step doesn't change it

$\Rightarrow$



$$\Rightarrow \quad Z_0 = Z_1 + \frac{1}{\left(\frac{1}{Z_2} + \frac{1}{Z_0}\right)}$$

$$= Z_1 + \frac{1}{\frac{(Z_0 + Z_2)}{Z_0 Z_2}}$$

$$= Z_1 + \frac{Z_0 Z_2}{Z_0 + Z_2}$$

Multiply by $Z_0 + Z_2$

$$Z_0 (Z_0 + \cancel{Z_2}) = Z_1 (Z_0 + Z_2) + \cancel{Z_0 Z_2}$$

$$Z_0^2 = Z_1 Z_0 + Z_1 Z_2$$

$$0 = Z_0^2 - Z_1 Z_0 - Z_1 Z_2,$$

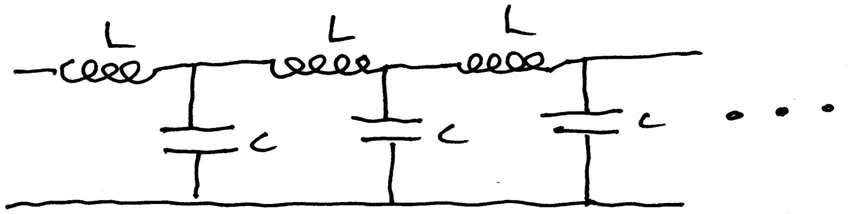
quadratic equation for Z_0

[Solution $0 = ax^2 + bx + c \Rightarrow x = \frac{1}{2}\left[-b \pm \sqrt{b^2 - 4ac}\right]$]

So $Z_0 = \frac{1}{2} Z_1 \pm \frac{1}{2}\sqrt{Z_1^2 + 4 Z_1 Z_2}$

$$= \frac{1}{2} Z_1 \pm \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}$$

e.g.



$Z_1 = i\omega L$, $Z_2 = \dfrac{1}{i\omega C}$

$$Z_0 = \frac{i\omega L}{2} + \sqrt{-\frac{\omega^2 L^2}{4} + \frac{L}{C}}$$

$$= \frac{i\omega L}{2} + \sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}}.$$

Two cases

If $\frac{L}{C} - \frac{\omega^2 L^2}{4} > 0$, the square root is real.

If $\frac{L}{C} - \frac{\omega^2 L^2}{4} < 0$, the square root is imaginary,

$$\sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}} = \sqrt{(-1)\underbrace{\left(\frac{\omega^2 L^2}{4} - \frac{L}{C}\right)}_{>0}}$$

$$= i \underbrace{\sqrt{\frac{\omega^2 L^2}{4} - \frac{L}{C}}}_{\text{real}}.$$

The transition between the two behaviours is the critical frequency ω_c,

$$\frac{L}{C} - \frac{\omega_c^2 L^2}{4} = 0$$

$$\boxed{\omega_c = \frac{2}{\sqrt{LC}}}$$

For $\omega < \omega_c$: $Z_0 = \frac{i\omega L}{2} + \sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}}$

↑ imaginary ↑ real.

For $\omega > \omega_c$: $Z_0 = \frac{i\omega L}{2} + i\sqrt{\frac{\omega^2 L^2}{4} - \frac{L}{C}}$

imaginary

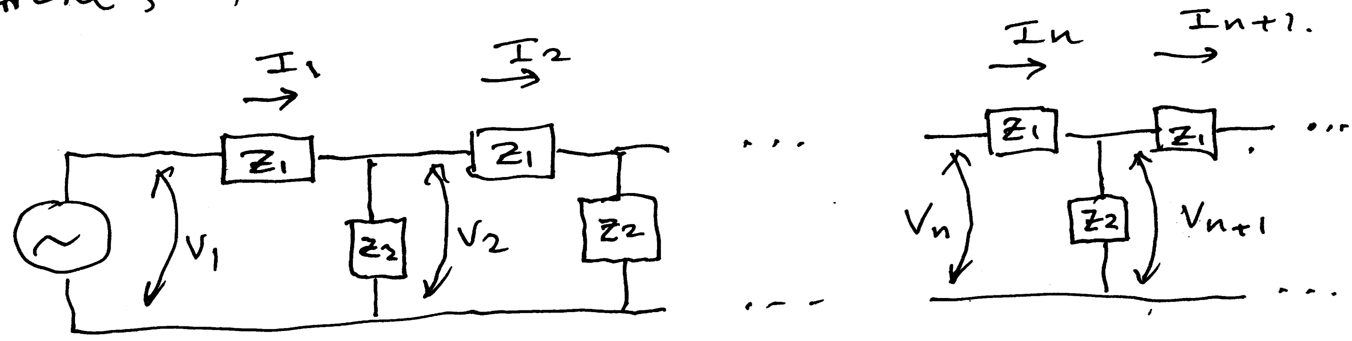
We saw yesterday that a circuit element with a purely imaginary impedance does not extract energy from the rest of the circuit, but if there is a real part to the impedance, energy is taken from the rest of the circuit.

So ~~for~~ $\omega < \omega_c$, energy will have to be pumped into the ladder network to keep it going. How can this be when the network is made only from inductors and capacitors that we saw are nondissipative (store energy but it is not "lost").

The answer is that when we place a voltage across the ladder, current will start flowing in the first stage of the ladder, then in the second stage, then in the third stage etc. Since the chain is infinitely long, We must continuously provide energy.

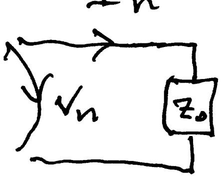
But for $\omega > \omega_c$, the impedance is purely imaginary $\Rightarrow$ zero power consumed on average. This can only be so if somehow the current dies off as we go further into the ladder (otherwise we would have to continuously provide energy). We will see below that this is indeed the case.

• Let's analyze the voltages and currents in the ladder network:



$$V_{n+1} = V_n - \underbrace{I_n Z_1}_{\text{voltage drop}}$$

But



$\leftarrow$ rest of ladder

$$\Rightarrow I_n = \frac{V_n}{Z_0}.$$

So $V_{n+1} = V_n - \frac{V_n}{Z_0} Z_1$

$$\Rightarrow \quad \frac{V_{n+1}}{V_n} = 1 - \frac{z_1}{Z_0} = \alpha .$$

called the propagation factor.

$$\text{So} \quad V_{n+1} = \alpha V_n$$
$$= \alpha^2 V_{n-1}$$
$$= \ldots$$
$$= \alpha^n V_1$$

applied voltage

- Let's compute α for the cases $\omega < \omega_c$ and $\omega > \omega_c$.

Recall

$$Z_0 = \begin{cases} \frac{i\omega L}{2} + \sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}} & \omega < \omega_c \\ \frac{i\omega L}{2} + i\sqrt{\frac{\omega^2 L^2}{4} - \frac{L}{C}} & \omega > \omega_c \end{cases}$$

$$z_1 = i\omega L$$

- For $\omega < \omega_c$:

$$\alpha = 1 - \frac{z_1}{Z_0}$$
$$= \frac{Z_0 - z_1}{Z_0}$$
$$= \frac{\frac{i\omega L}{2} + \sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}} - i\omega L}{\frac{i\omega L}{2} + \sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}}}$$

$$= \frac{\sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}} - \frac{i\omega L}{2}}{\sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}} + \frac{i\omega L}{2}}.$$

This is a complex number of the form $z = \frac{a - ib}{a + ib}$.

For such a number, $|z| = 1$

Check: $|z|^2 = z . z^*$

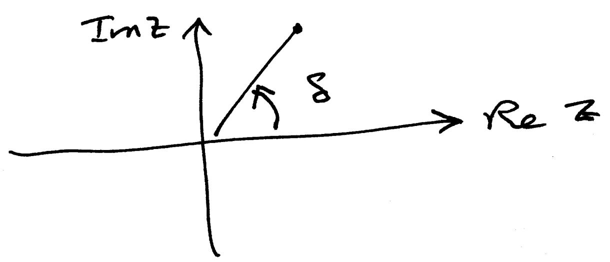
$$= \frac{a - ib}{a + ib} \; \frac{a + ib}{a - ib}$$

$$= \frac{a^2 + b^2}{a^2 + b^2}$$

$$= 1.$$

So α is a complex number with modulus 1

$\Rightarrow \quad \alpha = 1 . e^{i\delta} = e^{i\delta}$



So: with $V_{n+1} = \alpha V_n$,

$$|V_{n+1}| = |\alpha| \, |V_n|$$
$$= |V_n|$$

ie the magnitude of the voltage does not change as we move along the ladder, only the phase

$$V_n = |V_n| e^{i\omega t}$$

$$V_{n+1} = \alpha V_n = e^{i\delta} |V_n| e^{i\omega t}$$

$$= |V_n| e^{i(\omega t + \delta)}$$

This is in accord with what we expect: for $\omega < \omega_c$, the circuit will keep using energy as more and more parts become <u>active</u>.

- Now consider $\omega > \omega_c$

$$\alpha = \frac{Z_0 - Z_1}{Z_0}$$

$$= \frac{\frac{i\omega L}{2} + i\sqrt{\frac{\omega^2 L^2}{4} - \frac{L}{C}} - i\omega L}{\frac{i\omega L}{2} + i\sqrt{\frac{\omega^2 L^2}{4} - \frac{L}{C}}}$$

$$= \frac{\sqrt{\frac{\omega^2 L^2}{4} - \frac{L}{C}} - \frac{\omega L}{2}}{\sqrt{\frac{\omega^2 L^2}{4} - \frac{L}{C}} + \frac{\omega L}{2}}$$

This is a <u>real number</u> as the factors of i cancel in the numerator and denominator.

Further, $\alpha < 1$ as the numerator is smaller than the denominator

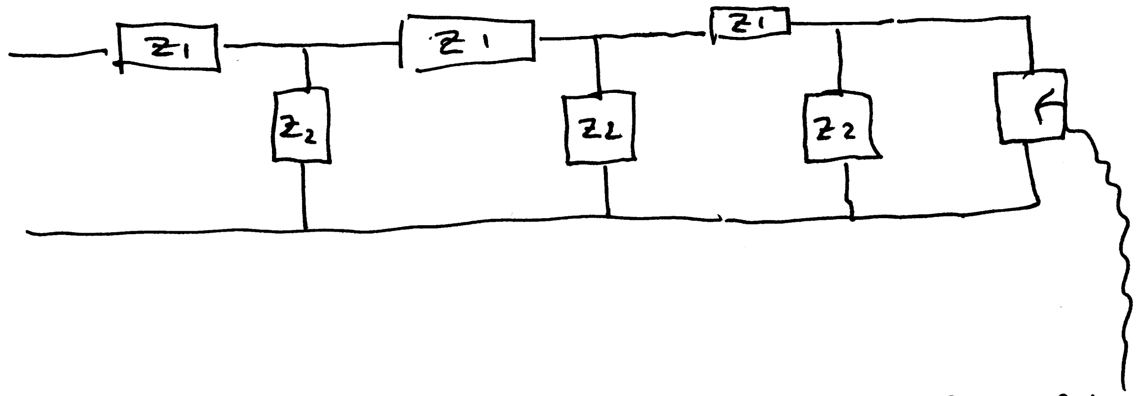
So $\alpha^n \to 0$ as $n \to \infty$ (e.g. $\alpha = 0.5$, $\alpha^2 = 0.25$, $\alpha^3 = 0.125$, $\alpha^4 = 0.0625$, ...).

So with $V_{n+1} = \alpha^n V_1$, the voltage in the circuit will drop off rapidly until it is effectively zero, the circuit will not allow any more current to flow $\Rightarrow$ energy is not needed continually.

- In fact, we can plot $|\alpha|$

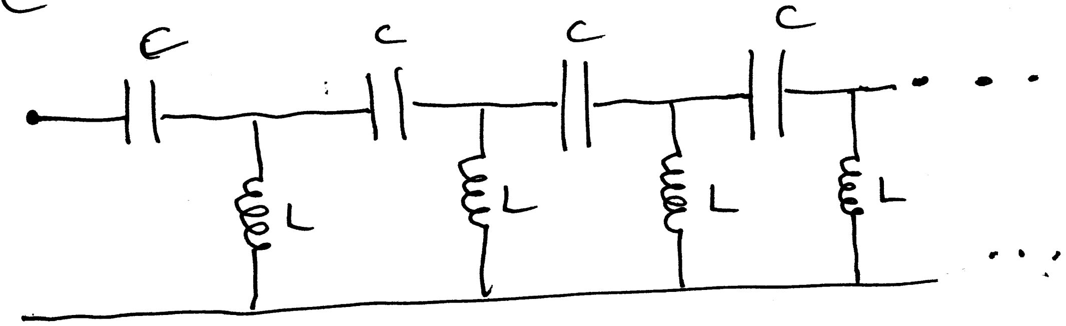
This is called a "low pass filter": signals with frequency $\omega < \omega_c$ free pass through the ladder network, signals with $\omega > \omega_c$ are rapidly attenuated and do not pass through the ladder network.

- Of course, we can't in practice build an infinite network. <u>But</u>: given that the impedance of the infinite network is Z_0, we can approximate it as



build a circuit element with an impedance $\propto Z_0$ to mimic the rest of the ladder.

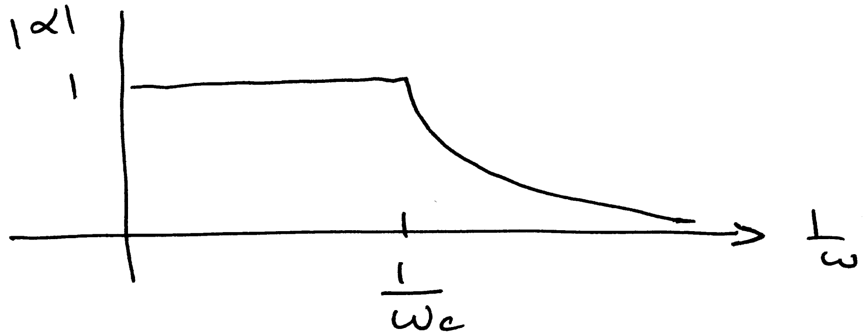
- What if we interchange L and C



In our calculation of α,
we can simply replace $i\omega L$ by
$\frac{1}{i\omega C}$ and vice-versa.

ie $L \Leftrightarrow C$, $i\omega \Leftrightarrow \frac{1}{i\omega}$.

So the dependence of α on ω
will become a dependence on $\frac{1}{\omega}$



So for $\frac{1}{\omega} < \frac{1}{\omega_c}$, no blocking of signals

ie $\omega > \omega_c \Rightarrow$ signals propagate freely with only a phase shift.

For $\frac{1}{\omega} > \frac{1}{\omega_c}$, signal rapidly attenuated

ie $\omega < \omega_c \Rightarrow$ signals <u>blocked</u>

This is called a <u>high pass</u> filter,
lets high frequency signals through.