EM EXAMINATION FORMULA SHEET

- The surface area of a sphere of radius R is $4\pi R^2$ and its volume is $\frac{4}{3}\pi R^3$.
- $\ln a - \ln b = \ln \frac{a}{b}$.
- $e^{i\theta} = \cos\theta + i\,\sin\theta$.
- Product rule: $\frac{d}{dx}\left(f(x)\,g(x)\right) = \frac{df(x)}{dx}\,g(x) + f(x)\,\frac{dg(x)}{dx}$.
- Chain rule: if $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, then
$$\frac{\partial}{\partial x}\,f(r) = \frac{df(r)}{dr}\,\frac{\partial r}{\partial x}.$$
- The unit vectors in the x, y and z directions are
$$\vec{e}_x = (1,0,0), \quad \vec{e}_y = (0,1,0), \quad \vec{e}_z = (0,0,1).$$
- A vector $\vec{A}$ with components (A_x, A_y, A_z) can be expressed in terms of the unit vectors $\vec{e}_x, \vec{e}_y$ and $\vec{e}_z$ as
$$\vec{A} = A_x\,\vec{e}_x + A_y\,\vec{e}_y + A_z\,\vec{e}_z.$$
- $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$.
- $\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\vec{e}_x + (A_zB_x - A_xB_z)\vec{e}_y + (A_xB_y - A_yB_x)\vec{e}_z$.
- The point with Cartesian coordinates (x, y, z) has a position vector
$$\vec{r} = x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z.$$
The length of the position vector is
$$r = |\vec{r}\,| = (x^2 + y^2 + z^2)^{\frac{1}{2}}.$$
- The unit vector in the radial direction is
$$\vec{e}_r = \frac{\vec{r}}{r}.$$
- $\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.
- The gradient of a scalar field $\phi(\vec{r}\,)$ is the vector field
$$\vec{\nabla}\,\phi(\vec{r}\,) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right).$$
- The divergence of a vector field $\vec{A}(\vec{r}\,)$ is the scalar field
$$\vec{\nabla}\cdot\vec{A}(\vec{r}\,) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

- The curl of a a vector field $\vec{A}(\vec{r})$ is the vector field

$$\vec{\nabla} \times \vec{A}(\vec{r}) = (\partial_y A_z - \partial_z A_y,\, \partial_z A_x - \partial_x A_z,\, \partial_x A_y - \partial_y A_x)\,.$$

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$$\phi(\vec{r} + d\vec{r}) = \phi(\vec{r}) + \vec{\nabla}\phi(\vec{r}) \cdot d\vec{r}.$$

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$$\begin{aligned} \vec{\nabla} \times \vec{\nabla}\phi(\vec{r}, t) &= 0 \\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}(\vec{r}, t)) &= 0. \end{aligned}$$

- Line integrals: if Γ is a curve from $\vec{r}_1$ to $\vec{r}_2$, then

$$\int_\Gamma \vec{\nabla}\phi(\vec{r}) \cdot d\vec{\ell} = \phi(\vec{r}_2) - \phi(\vec{r}_1)$$

- Stokes' theorem says that if S is any two-dimensional surface whose boundary is the closed curve Γ, then the circulation of a vector field $\vec{A}(\vec{r})$ around Γ is related to the flux of the curl of the vector field through the surface S:

$$\oint_\Gamma \vec{A}(\vec{r}) \cdot d\vec{\ell} = \int_S \left(\vec{\nabla} \times \vec{A}(\vec{r})\right) \cdot d\vec{S}$$

- Gauss's theorem relates the flux of a vector field through a closed two-dimensional surface S to the integral of the divergence of the vector field over the volume V enclosed by the surface:

$$\oint_S \vec{A}(\vec{r}) \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{A}(\vec{r})\, d^3\vec{r}$$

- Gauss's law: if V is a volume enclosed by a closed surface S, then

$$\oint_S \vec{E}(\vec{r}) \cdot d\vec{S} = \frac{Q}{\epsilon_0},$$

  where Q is the charge in the volume V.

- Maxwell's equations in general:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E}(\vec{r}, t) &= \frac{\rho(\vec{r}, t)}{\epsilon_0} \quad \text{(Gauss's law)} \\ \vec{\nabla} \times \vec{E}(\vec{r}, t) &= -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t} \quad \text{(Faraday's law)} \\ \vec{\nabla} \cdot \vec{B}(\vec{r}, t) &= 0 \\ \vec{\nabla} \times \vec{B}(\vec{r}, t) &= \frac{1}{c^2}\frac{\partial \vec{E}(\vec{r}, t)}{\partial t} + \frac{\vec{j}(\vec{r}, t)}{\epsilon_0 c^2} \quad \text{(Ampere's law)} \end{aligned}$$

- Lorentz force:

$$F = q\vec{E} + q\vec{v} \times \vec{B}.$$

- Relation between electric potential energy $U(\vec{r})$ and electric potential $V(\vec{r})$ for a charge q:

$$U(\vec{r}) = q\,V(\vec{r}).$$

- The electric field due to a point charge q at the origin is

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0 r^2}\,\vec{e}_r,$$

where $\vec{e}_r = \frac{\vec{r}}{r}$ is the unit vector in the radial direction.

- The electric potential due to a point charge q at the origin is

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}.$$

- Integral version of Gauss's law: if V is a volume enclosed by a closed surface S, then

$$\oint_S \vec{E}(\vec{r}) \cdot d\vec{S} = \frac{Q}{\epsilon_0},$$

where Q is the charge in the volume V.

- The current I through a surface S is

$$I = \int_S \vec{j}(\vec{r}) \cdot d\vec{S},$$

where $\vec{j}(\vec{r})$ is the current density (current per unit cross-sectional area).

- Integral version of Ampere's law in magnetostatics: if S is a two dimensional surface with boundary Γ,

$$\oint_\Gamma B(\vec{r}) \cdot d\vec{l} = \frac{I}{\epsilon_0 c^2},$$

where I is the current through the surface S.

- For static electric fields, the electric potential $V(\vec{r})$ and the electric field $\vec{E}(\vec{r})$ are related as follows:

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r}).$$

If Γ is any path from point $\vec{r}_1$ to point $\vec{r}_2$,

$$V(\vec{r}_2) - V(\vec{r}_1) = -\int_\Gamma \vec{E}(\vec{r}) \cdot d\vec{\ell}.$$

- The vector potential is defined by

$$\vec{B}(\vec{r}, t) = \vec{\nabla} \times \vec{A}(\vec{r}, t).$$

- Induced electromotive force (emf) in a circuit:

$$\mathcal{E} = -\frac{d\Phi(t)}{dt},$$

where $\Phi(t)$ is the magnetic flux through the circuit.

- Capacitance C: for a pair of conductors with charges $+Q$ and $-Q$ and with a potential difference ΔV, the capacitance C is defined by

$$\Delta V = \frac{1}{C}\,Q.$$

The potential energy contained in the electric fields in a capacitor is

$$U = \frac{1}{2}\,\frac{Q^2}{C} = \frac{1}{2}\,C\,(\Delta V)^2.$$

- Inductance L: for a circuit element with current I a magnetic flux Φ, the inductance L is defined by:

$$\Phi = L\,I.$$

The potential energy contained in the magnetic field in an inductor is

$$U = \frac{1}{2} L\,I^2.$$

The induced electromotive force (or "back emf") in an inductor is

$$\mathcal{E} = -L\,\frac{dI(t)}{dt}.$$

- The Poynting vector is

$$\vec{S}(\vec{r},t) = \epsilon_0\,c^2\,\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t).$$

Energy flows in the direction of $\vec{S}$, and $|\vec{S}|$ is the amount of energy per unit time (power) flowing per unit cross-sectional area.

- The impedance z of a circuit element is defined by

$$z = \frac{\delta V(t)}{I(t)},$$

where sinusoidally oscillating voltage sources and currents are expressed as complex exponentials.

- Impedances for circuit elements in an AC circuit with applied voltage $\Delta V = V_0\,e^{i\omega t}$:

$$\begin{aligned} z_{\mathrm{resistor}} &= R \\ z_{\mathrm{inductor}} &= i\omega\,L \\ z_{\mathrm{capacitor}} &= \frac{1}{i\omega\,C}. \end{aligned}$$

- $i = e^{i\frac{\pi}{2}}, \quad -i = e^{-i\frac{\pi}{2}}$

- For impedances in series,

$$z_{\mathrm{total}} = z_1 + z_2 + \cdots$$

For impedances in parallel,

$$\frac{1}{z_{\mathrm{total}}} = \frac{1}{z_1} + \frac{1}{z_2} + \cdots$$