EM PROBLEM SET 9

SOLUTIONS

1(a). On the inner sphere, the charge density is

$$-\frac{Q}{4\pi (R_1)^2}.$$

On the outer sphere, it is $\frac{+Q}{4\pi (R_2)^2}$.

(b). Gauss's law:

(flux of $\vec{E}$ field through a closed surface S)

$= \frac{1}{\varepsilon_0}$ (charge enclosed by surface).

Choose the Gaussian surface to be the surface of a sphere of radius r.
Since the electric field is radial in direction, flux through the surface of the sphere $= 4\pi r^2 \ E(r)$.

(i) For $r < R_1$, charge enclosed by a sphere of radius r is 0

$$\Rightarrow 4\pi r^2 E(r) = 0$$

$$\Rightarrow E(r) = 0$$

(ii) For $R_1 < r < R_2$, charge enclosed by a sphere of radius r is $-Q$

$$\Rightarrow 4\pi r^2 E(r) = -\frac{Q}{\varepsilon_0}$$

$$\Rightarrow E(r) = -\frac{Q}{4\pi\varepsilon_0 r^2}.$$

The minus sign means the field is pointing inwards.

(iii) For $r > R_2$, charge enclosed by a sphere of radius r is $(-Q) + Q = 0$

$$\Rightarrow 4\pi r^2 E(r) = 0$$

$$\Rightarrow \underline{E(r) = 0}$$

(c). The electric field between the conducting spherical surface points <u>inwards</u>

$\Rightarrow$ force on a charge q at a distance r from centre is $qE(r) = -q\frac{Q}{4\pi\varepsilon_0 r^2}$

$\Rightarrow$ a force $F(r) = +q \dfrac{Q}{4\pi\varepsilon_0 r^2}$

must be exerted to move the charge outwards.

Work done in moving distance dr is

$$F(r)\,dr = \frac{qQ}{4\pi\varepsilon_0 r^2}\,dr$$

$\rightarrow$ Work done in moving from $r = R_1$ to $r = R_2$ is

$$W = \int_{R_1}^{R_2} F(r)\,dr$$

$$= \int_{R_1}^{R_2} \frac{qQ}{4\pi\varepsilon_0} \frac{1}{r^2}\,dr$$

$$= \frac{qQ}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{R_1}^{R_2}$$

$$= \frac{qQ}{4\pi\varepsilon_0}\left[-\frac{1}{R_2} - \left(-\frac{1}{R_1}\right)\right]$$

$$= \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

(d) Electric potential difference

= work per unit charge to move from R_1 to R_2

$$= \frac{1}{q}\frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$C = \frac{Q}{\Delta V}$$

$$= \frac{Q}{\left(\frac{Q}{4\pi\varepsilon_0}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

$$= \frac{4\pi\varepsilon_0}{\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

(e) The energy density in the electric field is

$$u(r) = \frac{\varepsilon_0}{2} E(r)^2$$

$$= \frac{\varepsilon_0}{2} \frac{Q^2}{(4\pi\varepsilon_0)^2 r^4}$$

The energy in a spherical shell between r and $r + dr$ (which has volume $4\pi r^2 dr$) is

$$dU = 4\pi r^2 dr \frac{\varepsilon_0}{2} \frac{Q^2}{(4\pi\varepsilon_0)^2 r^4}$$

$$= \frac{1}{2} \frac{Q^2}{4\pi\varepsilon_0} \frac{dr}{r^2}$$

Integrating from $r = R_1$ to $r = R_2$

$$U = \int_{R_1}^{R_2} \frac{1}{2} \frac{Q^2}{4\pi\varepsilon_0} \frac{dr}{r^2}$$

$$= \frac{1}{2} \frac{Q^2}{4\pi\varepsilon_0} \left[-\frac{1}{r}\right]_{R_1}^{R_2}$$

$$= \frac{1}{2} \frac{Q^2}{4\pi\varepsilon_0} \left[-\frac{1}{R_2} - \left(-\frac{1}{R_1} \right) \right]$$

$$= \frac{1}{2} \frac{Q^2}{4\pi\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right).$$

On the other hand

$$\frac{1}{2} \frac{Q^2}{C_1}$$

$$= \frac{1}{2} Q^2 \frac{1}{\left(\frac{4\pi\varepsilon_0}{\left(\frac{1}{R_1} - \frac{1}{R_2} \right)} \right)}$$

$$= \frac{1}{2} \frac{Q^2}{4\pi\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \checkmark$$

2 (a) Choose an Amperian loop which is a circle of radius r centred on the centre of the torus.

The circulation of the $\vec{B}$ field around the loop is $2\pi r\, B(r)$.

Ampere's law says

(circulation of $\vec{B}$ around loop)

$= \frac{1}{\varepsilon_0 c^2}$ (current through the loop).

(i) For $r < R_1$, current through the circle of radius r is zero

$\Rightarrow \quad 2\pi r\, B(r) = 0$

$\Rightarrow \quad \underline{B(r) = 0}$

(ii) For $R_1 < r < R_2$, current through the Amperian loop is NI

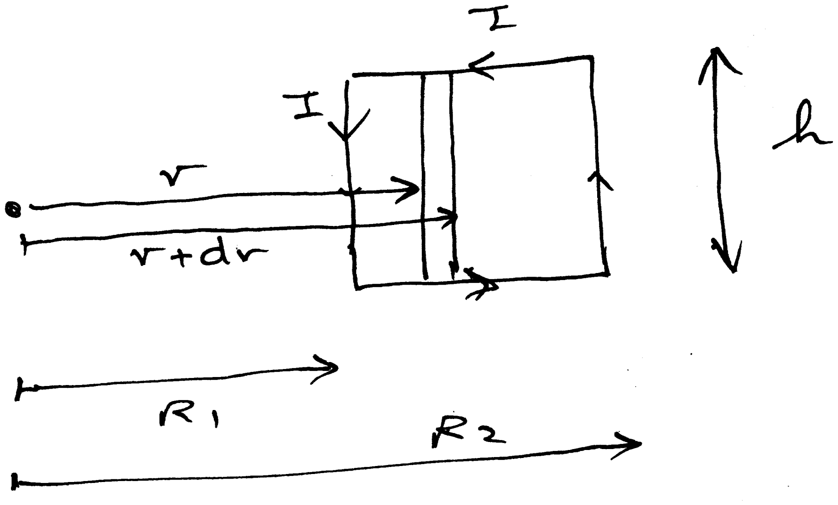
$$\Rightarrow \quad 2\pi r\, B(r) = \frac{NI}{\varepsilon_0 c^2}$$

$$\Rightarrow \quad B(r) = \frac{NI}{2\pi \varepsilon_0 c^2 r}$$

(ii) For $r > R_2$, current ~~through~~ through Amperian loop is $I + (-I) = 0$

$\Rightarrow 2\pi r\, B(r) = 0 \quad \Rightarrow \boxed{B(r) = 0}$

(b) Consider a cross-section through one loop of wire:



Magenetic flux through the vertical slice between r and $r+dr$

$$= B(r) \times \text{area}$$

$$= B(r) \times h\, dr$$

$$= \frac{NIh}{2\pi \varepsilon_0 c^2} \frac{dr}{r}$$

Integrating from $r = R_1$ to $r = R_2$,
Magnetic flux through one turn of wire

$$= \int_{R_1}^{R_2} \frac{NIh}{2\pi \varepsilon_0 c^2} \frac{dr}{r} = \frac{NIh}{2\pi \varepsilon_0 c^2} \left[\ln r\right]_{R_1}^{R_2}$$

$$= \frac{NIh}{2\pi\varepsilon_0 c^2} (\ln R_2 - \ln R_1)$$

$$= \frac{NIh}{2\pi\varepsilon_0 c^2} \ln\left(\frac{R_2}{R_1}\right).$$

(c) Total flux through toroid

$= N \times$ (flux through one loop)

$$= \frac{N^2 I h}{2\pi\varepsilon_0 c^2} \ln \frac{R_2}{R_1}$$

$\Rightarrow$ Inductance

$$L = \frac{\Phi}{I}$$

$$= \frac{N^2 h}{2\pi\varepsilon_0 c^2} \ln \frac{R_2}{R_1}$$

(d). Energy density

$$u(r) = \frac{1}{2} \varepsilon_0 c^2 \; |B(r)|^2$$

$$= \frac{1}{2} \varepsilon_0 c^2 \; \frac{N^2 I^2}{(2\pi\varepsilon_0 c^2)^2 \, r^2}$$

For a strip between r and $r + dr$,

volume $= 2\pi r \, . \, dr \, . \, h$

$\Rightarrow$ energy in the strip is

$$dU = 2\pi r\, dr\, h\, \frac{1}{2}\varepsilon_0 c^2 \frac{N^2 I^2}{(2\pi\varepsilon_0 c^2)^2 r^2}$$

$$= \frac{1}{2}\frac{N^2 I^2 h}{2\pi\varepsilon_0 c^2}\frac{dr}{r}$$

$$\Rightarrow U = \int_{R_1}^{R_2} dU$$

$$= \frac{1}{2}\frac{N^2 I^2 h}{2\pi\varepsilon_0 c^2}\int_{R_1}^{R_2}\frac{dr}{r}$$

$$= \frac{1}{2}\frac{N^2 I^2 h}{2\pi\varepsilon_0 c^2}\left[\ln r\right]_{R_1}^{R_2}$$

$$= \frac{1}{2}\frac{N^2 I^2 h}{2\pi\varepsilon_0 c^2}\left(\ln R_2 - \ln R_1\right)$$

$$= \frac{1}{2}\frac{N^2 I^2 h}{2\pi\varepsilon_0 c^2}\ln\frac{R_2}{R_1}$$

(e) $\frac{1}{2} L I^2$

$$= \frac{1}{2}\frac{N^2 h}{2\pi\varepsilon_0 c^2}\ln\left(\frac{R_2}{R_1}\right) I^2$$

$$= U \quad \checkmark$$