Energy transport by electromagnetic waves

For plane polarised EM waves,

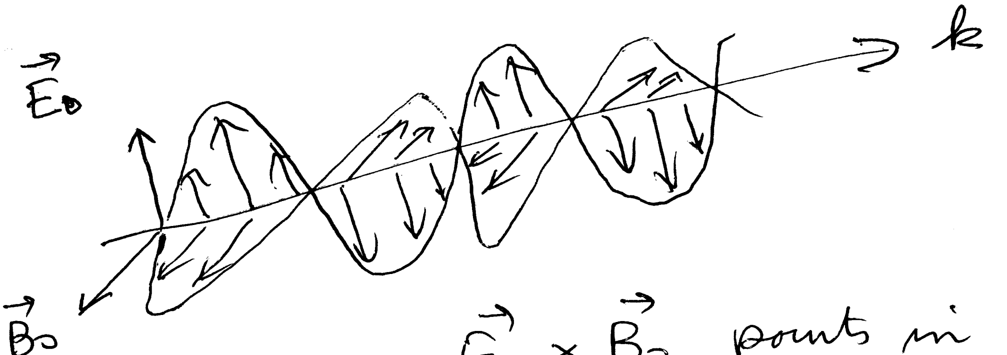
$$\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}.\vec{r} - \omega t)$$

$$\vec{B}(\vec{r},t) = \vec{B}_0 \cos(\vec{k}.\vec{r} - \omega t)$$

- with $\frac{\omega}{|\vec{k}|} = c =$ speed of light.

$\vec{k}$ points in the direction of the propagation of the waves, and the wavelength is $\lambda = \frac{2\pi}{|\vec{k}|}$.

- $\vec{E}_0$ and $\vec{B}_0$ are perpendicular to $\vec{k}$ and to each other, and $|\vec{B}_0| = \frac{1}{c}|\vec{E}_0|$



$\vec{E}_0 \times \vec{B}_0$ points in direction $\vec{k}$ of propagation.

Then the Poynting vector is

$$\vec{S}(\vec{r},t) = \varepsilon_0 c^2 \vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t)$$

$$= \varepsilon_0 c^2 \vec{E}_0 \times \vec{B}_0 \underbrace{\cos^2(\vec{k}\cdot\vec{r} - \omega t)}_{\geq 0}$$

So the flow of energy is in the direction $\vec{E}_0 \times \vec{B}_0$, which is the same as the direction of propagation of the wave.

Since $\vec{E}_0 \perp \vec{B}_0$

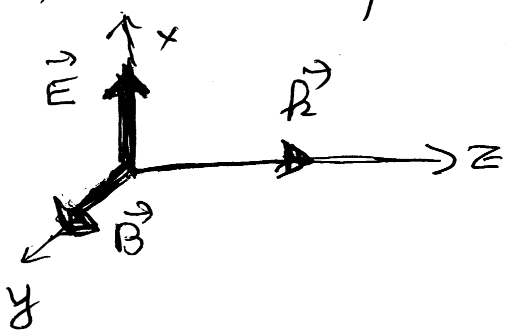
$$|\vec{E}_0 \times \vec{B}_0| = |\vec{E}_0| |\vec{B}_0|$$

$$= \frac{1}{c} |\vec{E}_0|^2$$

So the energy per unit time ($\equiv$ power) crossing a unit area perpendicular to $\vec{k}$ is $|\vec{S}| = \varepsilon_0 c^2 \frac{1}{c} |\vec{E}_0|^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$

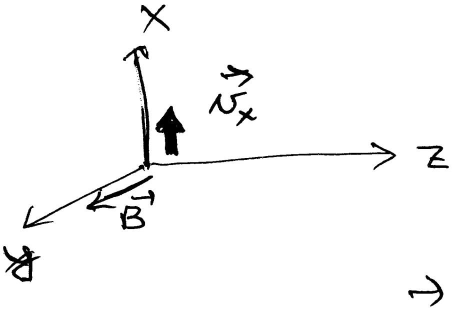
Electromagnetic waves also carry momentum

To demonstrate this, consider a particle of charge q in an electromagnetic field propagating along the z-axis, with the $\vec{E}$ field along the x-axis and the $\vec{B}$ field along the y axis

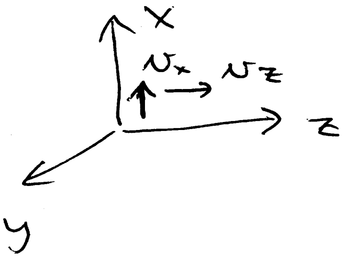


The particle will experience a Lorentz force $\vec{F} = q\vec{E} + q\,\vec{v} \times \vec{B}$.

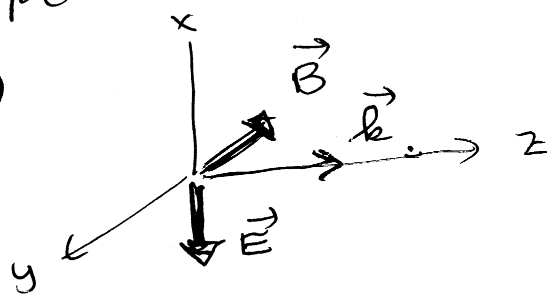
If the particle is initially at rest at the origin, $q\vec{v} \times \vec{B} = 0$ and it will start to move in the direction of the $\vec{E}$ field due to $F = q\vec{E}$ ie in the $+x$ direction



But now $q\vec{v}\times\vec{B}\neq 0$, it points in the $+Z$ direction, so the particle also starts to move in the $+Z$ direction.

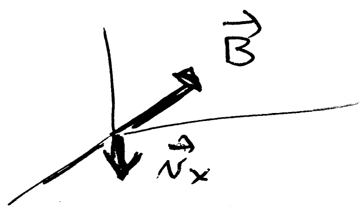


- The $\vec{E}$ and $\vec{B}$ fields are oscillating in phase, and they will also point as follows:

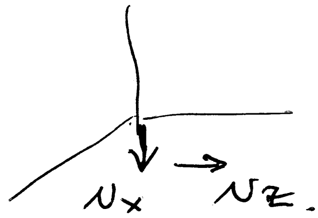


($\vec{E}\times\vec{B}$ is still in the direction $\vec{k}$).

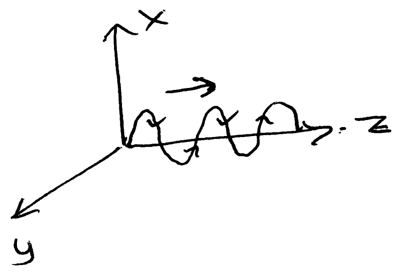
Now the particle will move in the $-x$ direction due to the $\vec{E}$ field,



But $q\vec{v}\times\vec{B}$ <u>still</u> points in the $+Z$ direction and so the particle is still experience a force in the $+z$ direction due to the $\vec{B}$ field



- So whilst the force in the x direction due to the $\vec{E}$ field oscillates up and down, the $q\vec{v}\times\vec{B}$ force in the z direction is <u>always</u> in the $+z$ direction. So the particle does this



- The average momentum of the particle in the x direction is zero, but it <u>gains</u> momentum all the time from the $\vec{B}$ field in the $+z$ direction.

This proves that EM waves carry momentum - the momentum of the charged particle in the $+z$ direction is provided by the EM field.

Relationship between energy and momentum flow for EM waves

- Energy flow is described by the Poynting vector $\vec{S} = \varepsilon_0 c^2 \vec{E} \times \vec{B}$
  – it is the energy per unit time (or power) passing through a unit cross-sectional area (so power per unit area).

- We can relate this to the flow of momentum by appealing to quantum mechanics.

Quantum mechanics says that EM waves are really streams of photons travelling in the direction of propagation of the EM wave. Photons are particles with zero rest mass that travel at the speed of light.

For a ~~p~~ free particle of rest mass m_0, special relativity tells us that if its momentum is $\vec{p}$, its energy is $E = \sqrt{\vec{p}\cdot\vec{p}\,c^2 + m_0^2 c^4}$.

(in the case of nonrelativistic particles, $\frac{v}{c} << 1$, $\sqrt{\vec{p}\cdot\vec{p}\,c^2 + m_0^2 c^4}$

$$\approx \underbrace{m_0 c^2}_{\uparrow \text{ rest energy}} + \underbrace{\tfrac{1}{2} m v^2}_{\uparrow \text{ kinetic energy}}).$$

For a photon, $m_0 = 0$

$$\Rightarrow \quad E = \sqrt{|\vec{p}|^2 c^2}$$

$$= |\vec{p}|c$$

$$\Rightarrow \quad |\vec{p}| = \frac{1}{c} E.$$

So in an EM wave: rate of flow of momentum per unit time per unit cross-sectional area

$= \frac{1}{c}$ (rate of flow of energy per unit time per unit cross-sectional area)

$= \frac{1}{c} \vec{S}$,

where $\vec{S} = \varepsilon_0 c^2 \vec{E} \times \vec{B}$ is the Poynting vector.

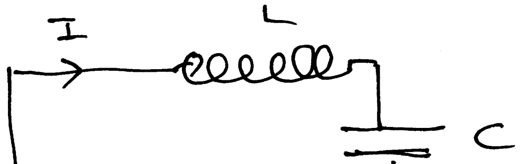
Note that energy and momentum flow are in the same direction, as we saw in the example of a charged particle in an EM field.

ALTERNATING CURRENT (AC) CIRCUITS

- These are circuits with sinusoidally varying emf's driving them, which in turn means sinusoidally varying currents.

A major element of the analysis of these circuits is that the <u>phase</u> of the applied emf and the resulting current will in general <u>differ</u>. We will see that use of notation using <u>complex numbers</u> easily allows us to keep track of the information about phases.

- For example, we have already seen that for an LC circuit with a current I established in it:



The charge on the capacitor Q(t) takes

the form

$$Q(t) = Q_0 \cos(\omega t + \phi)$$

$$\omega = \frac{1}{\sqrt{LC}}.$$

Then $I(t) = \dfrac{dQ(t)}{dt}$

$$= -\omega Q_0 \sin(\omega t + \phi).$$

$$= \omega Q_0 \cos(\omega t + \phi + \tfrac{\pi}{2})$$

- So the current is $\frac{\pi}{2} = 90^\circ$ ahead of the charge in phase. To work this out, we have to remember

$$\frac{d}{dt} \cos(\omega t + \phi) = -\omega \sin(\omega t + \phi)$$

and $-\sin(\omega t + \phi) = \cos(\omega t + \phi + \frac{\pi}{2})$.

- . But if we use complex notation

$Q(t) = Q_0 e^{i\omega t}$ [with the understanding that we take the real part,

$$\mathrm{Re}\,(Q_0 e^{i\omega t}) = \mathrm{Re}\cdot Q_0(\cos\omega t + i \sin \omega t)$$
$$= Q_0 \cos \omega t \quad].$$

Then $I(t) = \dfrac{dQ(t)}{dt}$

$$= i\omega Q_0 e^{i\omega t}$$

$$= e^{i\frac{\pi}{2}} \omega Q_0 e^{i\omega t}$$ (using $i = e^{i\frac{\pi}{2}}$, see below)

$$= \omega Q_0 e^{i(\omega t + \frac{\pi}{2})}.$$

- Taking the real part

$$I(t) = \omega Q_0 \underline{\cos(\omega t + \tfrac{\pi}{2})},$$

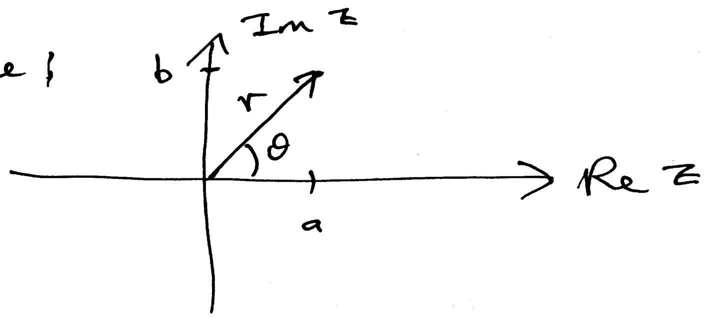
the same result as earlier but much more simple to derive.

Phasors

- Any complex number z can be represented in the form

$$z = \underset{\uparrow \atop Re(z)}{a} + i \underset{\uparrow \atop Im(z)}{b}$$

An alternative representation is possible;



$$z = \underset{\sqrt{a^2+b^2}}{\underset{\uparrow}{r}}\, e^{i\theta} \leftarrow \text{"phase factor"}$$

Proof: $r e^{i\theta}$

$= r\cos\theta + i r \sin\theta$

$= a + i b$

using basic trigonometry on the diagram.

- $z = r e^{i\theta}$ is called a "phasor".

θ contains phase information (in that it determines the phase of the cosine and sine in $e^{i\theta} = \cos\theta + i\sin\theta$).

- Differentiation

$$\frac{d}{dt}\cos\omega t = -\omega \sin\omega t$$

$$\frac{d}{dt}\sin\omega t = \omega\cos\omega t.$$

Rather than having to remember these rules, we can easily "encode" them using complex notation:

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

Differentiate w.r.t. t,

$$i\omega\, e^{i\omega t} = \frac{d}{dt}\cos\omega t + i\frac{d}{dt}\sin\omega t$$

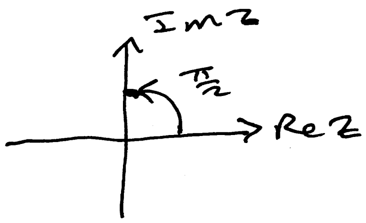
ie $$i\omega(\cos\omega t + i\sin\omega t) = \frac{d}{dt}\cos\omega t + i\frac{d}{dt}\sin\omega t.$$

Comparing LHS and RHS:

- No i: $$-\omega\sin\omega t = \frac{d}{dt}\cos\omega t$$

Factor of i: $$\omega\cos\omega t = \frac{d}{dt}\sin\omega t.$$

Phase shifts



- $$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i.1 = i$$

So $i = e^{i\frac{\pi}{2}}$, $-i = \frac{1}{i} = e^{-i\pi/2}$

Then $$i e^{i\omega t} = i(\cos\omega t + i\sin\omega t) = i\cos\omega t - \sin\omega t. \quad (1)$$

Alternatively $i e^{i\omega t}$
$$= e^{i\frac{\pi}{2}} e^{i\omega t}$$

$$= e^{i(\omega t + \frac{\pi}{2})}$$

$$= \cos(\omega t + \tfrac{\pi}{2}) + i \sin(\omega t + \tfrac{\pi}{2}) \quad \text{L}(2)$$

Comparing the real and imaginary parts of (1) and (2)

$$-\sin \omega t = \cos(\omega t + \tfrac{\pi}{2})$$
$$\cos \omega t = \sin(\omega t + \tfrac{\pi}{2})$$

↑ phase shifts

SUMMARY: complex notation allows us to very easily deal with derivatives of sinusoidal quantities and with phase shifts.

$$\boxed{\begin{array}{l} \text{Multiplication by } i \equiv \text{phase shift of } +\frac{\pi}{2} \\ \text{Multiplication by } -i \equiv \text{phase shift of } -\frac{\pi}{2} \end{array}}$$

Impedance

- For a resistor, the relationship between the applied voltage ΔV (or voltage drop across the resistor) and the current I is

$$\Delta V = I R$$

$$\Rightarrow \frac{\Delta V}{I} = R.$$

Since R is <u>real</u>, it means the phase of the applied voltage + the resulting current are the same:

$$\text{if } \Delta V = V_0 e^{i\omega t}$$

$$I = \frac{1}{R} \Delta V = \frac{V_0}{R} e^{i\omega t}$$

same phase

Taking real parts

$$\Delta V = V_0 \cos \omega t$$

$$I = \frac{V_0}{R} \cos \omega t.$$

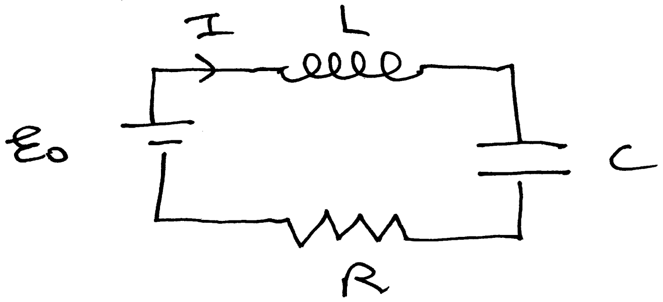
- For more general circuit elements (inductors, capacitors), ΔV and I are <u>not</u> in phase. So we have situations where

$$\Delta V = V_0 \, e^{i\omega t}$$
$$I = I_0 \, e^{i(\omega t + \phi)}$$

↑ phase shift by ϕ

$$\Rightarrow \frac{\Delta V}{I} = \frac{V_0 \, e^{i\omega t}}{I_0 \, e^{i(\omega t+\phi)}}$$
$$= \frac{V_0}{I_0} \, e^{-i\phi}$$

- In a circuit with an emf $\mathcal{E}_0$ and L, C, R:



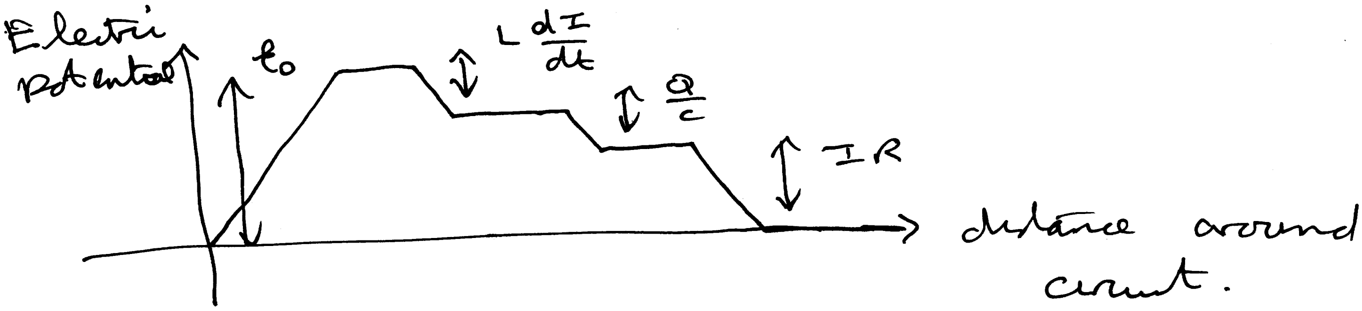
work done to create heat in resistor, potential energy of charge drops, $\Delta V = IR$

- $$\mathcal{E}_0 - L\frac{dI}{dt} - \frac{Q}{C} - IR = 0$$

work is done to increase the potential energy of a charge in the circuit (e.g battery)

work must be done to build up current $\Rightarrow$ potential energy of a charge drops $\Delta V = L\frac{dI}{dt}$

work must be done to build up charge on capacitor $\Rightarrow$ potential energy of a charge drops, $\Delta V = \frac{Q}{C}$



<u>So</u> $(\Delta V)_{\text{resistor}} = IR$

$(\Delta V)_{\text{inductor}} = L\dfrac{dI(t)}{dt}$

$(\Delta V)_{\text{capacitor}} = \dfrac{Q(t)}{C}$ ($Q(t)$ = charge on capacitor).

- In general, <u>impedance</u> is defined by

$$\boxed{Z = \frac{\Delta V}{I}}$$

- For a resistor $\Delta V = IR$

$$\Rightarrow Z_{\text{resistor}} = \frac{\Delta V}{I} = R \quad (\underline{\text{real}})$$

$$\boxed{Z_{\text{resistor}} = R}$$

- For an inductor

$$\Delta V = L\frac{dI}{dt}$$

If $I = I_0 e^{i\omega t}$

$$\Delta V = L i\omega I_0 e^{i\omega t}$$

$$\Rightarrow Z_{\text{inductor}} = \frac{\Delta V}{I} = \frac{iL\omega I_0 e^{i\omega t}}{I_0 e^{i\omega t}} = i\omega L$$

$$\boxed{Z_{\text{inductor}} = i\omega L}$$

- For a capacitor :

$$\Delta V = \frac{Q}{C}$$

$$\Rightarrow \quad I = \frac{dQ}{dt} = C \frac{d\,\Delta V}{dt}$$

If $\Delta V = V_0 e^{i\omega t}$

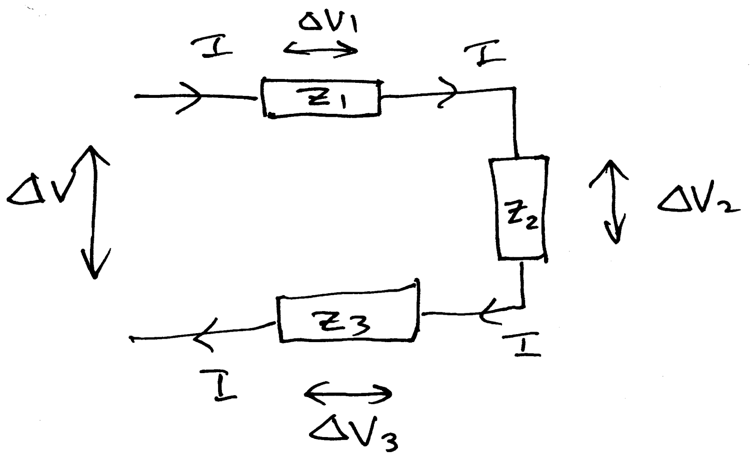
$$\frac{d\,\Delta V}{dt} = i\omega V_0 e^{i\omega t}$$

- $\Rightarrow$
$$Z_{capacitor} = \frac{\Delta V}{I} = \frac{V_0 e^{i\omega t}}{i\omega C V_0 e^{i\omega t}} = \frac{1}{i\omega C}$$

$$\boxed{Z_{capacitor} = \frac{1}{i\omega C}}$$

-

Impedances in series



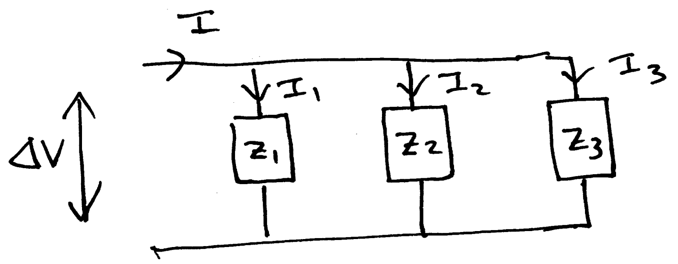
$$\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3$$
$$= I Z_1 + I Z_2 + I Z_3$$
$$= I(Z_1 + Z_2 + Z_3)$$

$$\Rightarrow Z_{total} = \frac{\Delta V}{I} = Z_1 + Z_2 + Z_3.$$

Impedances add when in series.

•

Impedance in parallel



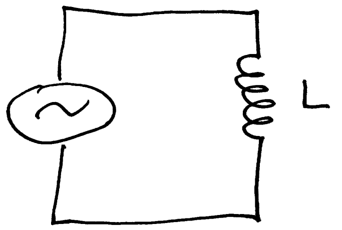
ΔV is the same for all three circuit elements.

• $I = I_1 + I_2 + I_3$

$$\Rightarrow \frac{1}{Z} = \frac{I}{\Delta V} = \frac{I_1}{\Delta V} + \frac{I_2}{\Delta V} + \frac{I_3}{\Delta V}$$
$$= \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

So inverse impedances add in parallel

Driven Inductive Circuit



$Z_{\text{inductor}} = i\omega L$

$$
\begin{aligned}
I &= \frac{V}{Z} \\
&= \frac{V}{i\omega L} \\
&= -\frac{i}{\omega L} V_0 e^{i\omega t} \\
&= \frac{e^{-i\frac{\pi}{2}}}{\omega L} V_0 e^{i\omega t} \\
&= \frac{V_0}{\omega L} e^{i(\omega t - \frac{\pi}{2})}
\end{aligned}
$$

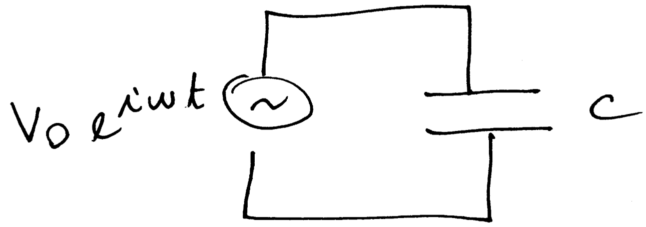
Taking <u>real</u> parts

$$V = V_0 \cos \omega t$$

$$I = \frac{V_0}{\omega L} \cos(\omega t - \tfrac{\pi}{2})$$

phase shift of $-\frac{\pi}{2} = -90^\circ$ between applied voltage and resulting current.

Driven Capacitive Circuit



$$Z_{\text{capacitor}} = \frac{1}{i\omega C}$$

$$I = \frac{V}{Z}$$

$$= \frac{V}{\left(\frac{1}{i\omega C}\right)}$$

$$= i\omega C V$$

$$= e^{i\frac{\pi}{2}} \omega C V_0 e^{i\omega t}$$

$$= \omega C V_0 e^{i(\omega t + \frac{\pi}{2})}$$

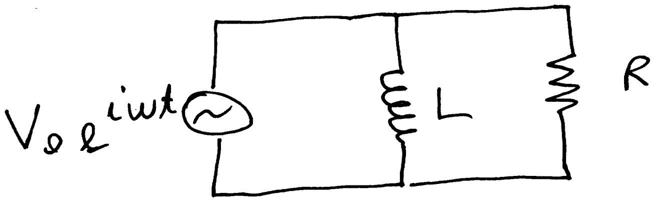
Taking real parts:

$$V = V_0 \cos \omega t$$

$$I = \omega C V_0 \cos(\omega t + \tfrac{\pi}{2})$$

phase shift of $\frac{\pi}{2} = 90°$, current "leads" voltage by $90°$

Parallel LR circuit



$$\frac{1}{Z_{total}} = \frac{1}{Z_{inductor}} + \frac{1}{Z_{resistor}}$$

$$= \frac{1}{i\omega L} + \frac{1}{R}$$

$$\Rightarrow \quad I = \frac{V}{Z_{total}}$$

$$= \frac{V_0 e^{i\omega t}}{i\omega L} + \frac{V_0 e^{i\omega t}}{R}$$

$$= -\frac{i}{\omega L} V_0 e^{i\omega t} + \frac{1}{R} V_0 e^{i\omega t}$$

$$= \frac{V_0}{\omega L}(-i)(\cos\omega t + i \sin\omega t)$$

$$+ \frac{V_0}{R}(\cos\omega t + i \sin\omega t)$$

Taking the real part

$$I = \frac{V_0}{\omega L} \sin\omega t + \frac{V_0}{R}\cos\omega t$$

current through inductor

current through resistor

90° out of phase

Alternatively:

$$I = -\frac{i}{\omega L} V_0 e^{i\omega t} + \frac{1}{R} V_0 e^{i\omega t}$$

$$= \frac{V_0}{\omega L} e^{-i\frac{\pi}{2}} e^{i\omega t} + \frac{V_0}{R} e^{i\omega t}$$

$$= \underbrace{\frac{V_0}{\omega L} e^{i(\omega t - \frac{\pi}{2})}}_{\text{current through inductor}} + \underbrace{\frac{V_0}{R} e^{i\omega t}}_{\text{current through resistor}}$$

current through inductor lags by $\frac{\pi}{2} = 90^\circ$

For $\omega L >> R$: current through inductor is very small compared to current through resistor.

Reason : ω large or L ⇒ current changing fast or inductance large ⇒ back emf in the inductor $L \frac{dI}{dt}$ is large ⇒ current "prefers" to flow through the resistor.

• So : if we have an applied voltage that contains a linear combination of a range of frequencies, this is a way to ensure the high frequency component passes through R (which might represent a load).

Note: we can also write Z in polar form

$$\frac{1}{Z} = \frac{1}{i\omega L} + \frac{1}{R}$$

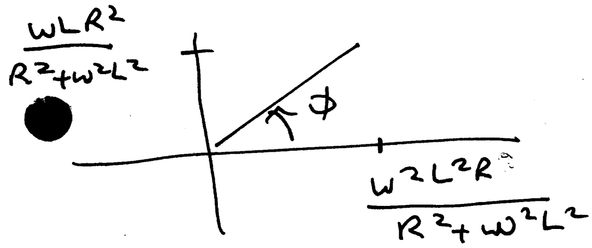
$$= \frac{R + i\omega L}{i\omega L R}$$

$$Z = \frac{i\omega L R}{R + i\omega L}$$

$$= \frac{i\omega L R \, (R - i\omega L)}{R^2 + \omega^2 L^2}$$

$$= \frac{\omega^2 L^2 R + i \omega L R^2}{R^2 + \omega^2 L^2}$$

$$= |Z| \, e^{i\phi}$$



$$|Z|^2 = \frac{\omega^2 L^2 R^4}{(R^2+\omega^2 L^2)^2} + \frac{\omega^4 L^4 R^2}{(R^2+\omega^2 L^2)^2}$$

$$= \frac{\omega^2 L^2 R^2}{(R^2+\omega^2 L^2)^2} (R^2 + \omega^2 L^2)$$

$$= \frac{\omega^2 L^2 R^2}{(R^2+\omega^2 L^2)}$$

$$|Z| = \frac{\omega L R}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\tan \phi = \frac{R}{\omega L}$$