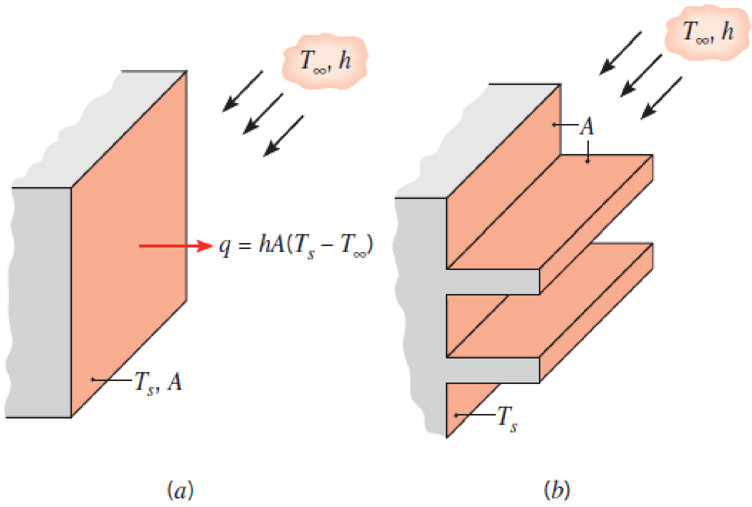
Heat Transfer from Extended Surfaces

2. 2. Fin Performance

- The fin itself represents a conduction resistance to heat transfer from the original surface.
- Accordingly, not always the extended surface enhances the heat transfer.
- The fin effectiveness (ε_f) is needed to evaluate the heat transfer through the extended surface.
- ε_f is defined as the ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin.



$$\varepsilon_f = \frac{q_f}{hA_{c,b}\theta_b}$$

where $A_{c,b}$ is the fin cross-sectional area at the base.

Heat Transfer from Extended Surfaces

2. 2. Fin Performance

- The value of ε_f should be as large as possible, and in general, the use of fins may rarely be justified unless $\varepsilon_f > 2$.
- Based on the boundary conditions used for the investigated system, the effectiveness for a fin of uniform cross section may be calculated.
- assuming the convection coefficient of the finned surface to be equivalent to that of the base surface, it follows that, for the infinite fin approximation (Case D), the result is:

$$\varepsilon_f = \left(\frac{kP}{hA_c}\right)^{1/2}$$

TABLE 3.4 Temperature distribution and heat loss for fins of uniform cross section

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b		Fin Heat Transfer Rate q_f	
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$	e^{-mx}	(3.84)	M	(3.85)

$\theta \equiv T - T_\infty$ $\qquad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\qquad M \equiv \sqrt{hPkA_c}\theta_b$

Heat Transfer from Extended Surfaces

2. 2. Fin Performance

➢ According to this equation, one can conclude that the following are needed achieving a fin-enhanced heat transfer:

1. *High thermal conductivity (Cu or Al).*
2. *High ratio of the perimeter to the cross-sectional area: thin fins*
3. *Fluids with low convection coefficient (gases).*

$k_{Cu} = 398$ W/m.K
$k_{Al} = 180$ W/m.K
$k_{Fe} = 14$ W/m.K

TABLE 1.1 Typical values of the convection heat transfer coefficient

Process	h (W/m$^2 \cdot$ K)
Free convection	
Gases	2–25
Liquids	50–1000
Forced convection	
Gases	25–250
Liquids	100–20,000

$$\varepsilon_f = \left(\frac{kP}{hA_c}\right)^{1/2}$$

$$L = 2.65/m$$

Heat Transfer from Extended Surfaces

2. 2. Fin Performance

- Fin performance may also be quantified in terms of a thermal resistance.
- The *fin resistance* may be defined as:

$$R_{t,f} = \frac{\theta_b}{q_f}$$

- For the bas surface without fin, then, convection resistance will be considered:

$$R_{t,b} = \frac{1}{hA_{c,b}}$$

- Using those equations, the effectiveness can be then determined:

$$\varepsilon_f = \frac{R_{t,b}}{R_{t,f}}$$

- To increase ε_f it is necessary to reduce ***the conduction/convection resistance*** of the fin.

$$\varepsilon_f = \left(\frac{kP}{hA_c}\right)^{1/2}$$

Heat Transfer from Extended Surfaces

2. 2. Fin Performance

- Another measure of fin thermal performance is provided by the fin efficiency (η_f).

$$\eta_f \equiv \frac{q_f}{q_{\max}} = \frac{q_f}{hA_f\theta_b}$$

- For a straight fin of uniform cross section and an adiabatic tip:

$$\eta_f = \frac{M \tanh mL}{hPL\theta_b} = \frac{\tanh mL}{mL}$$

- This result shows that η_f approaches its maximum and minimum values of 1 and 0, respectively, as L approaches 0 and infinity.

Heat Transfer from Extended Surfaces

2. 2. Fin Performance

Example-1

A stainless-steel pin fin that has k =16 W/m.K, L = 10 cm, d = 1 cm and that is exposed to a boiling-water convection situation with h = 5000 W/m^2.K. Based on these conditions, show whether this fin can enhance the heat transfer.

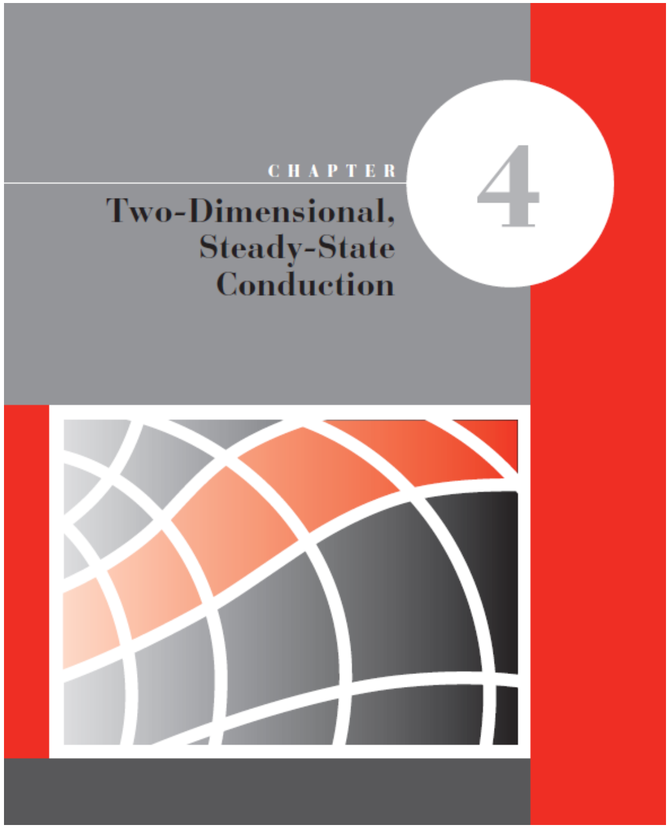
$$\frac{q \text{ with fin}}{q \text{ without fin}} = \frac{\tanh mL}{\sqrt{hA/kp}}$$

$$= \frac{\tanh\left\{\left[\frac{5000\pi(1\times10^{-2})(4)}{16\pi(1\times10^{-2})^2}\right]^{1/2}(10\times10^{-2})\right\}}{\left[\frac{5000\pi(1\times10^{-2})^2}{(4)(16)\pi(1\times10^{-2})}\right]^{1/2}}$$

$$= 1.13$$

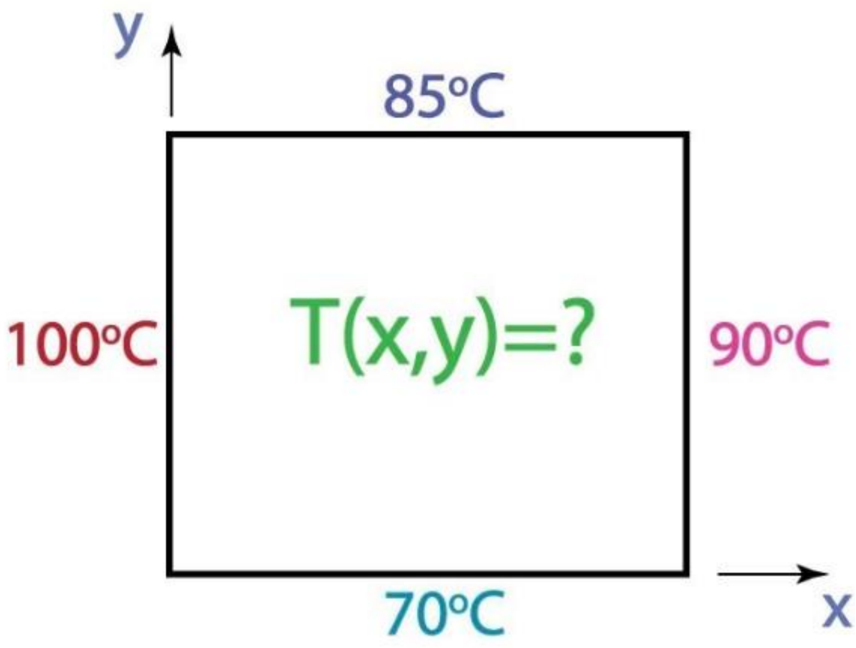
Thus, this rather large pin produces an increase of only 13 percent in the heat transfer.

Two-Dimensional, Steady-State Conduction



Two-Dimensional, Steady-State Conduction

- Up to this point, conduction problems in which the temperature gradient is significant for only one coordinate direction were considered.
- For more practical applications, multidimensional effects should be taken into account.



Two-Dimensional, Steady-State Conduction

1. Alternative Approaches

- Considering a prismatic solid in which there is two-dimensional heat conduction.
- Two parallel surfaces are insulated at the same temperature, and the other surface were maintained at different temperatures (temperature gradient).
- Considering, a steady state flow without energy generation.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

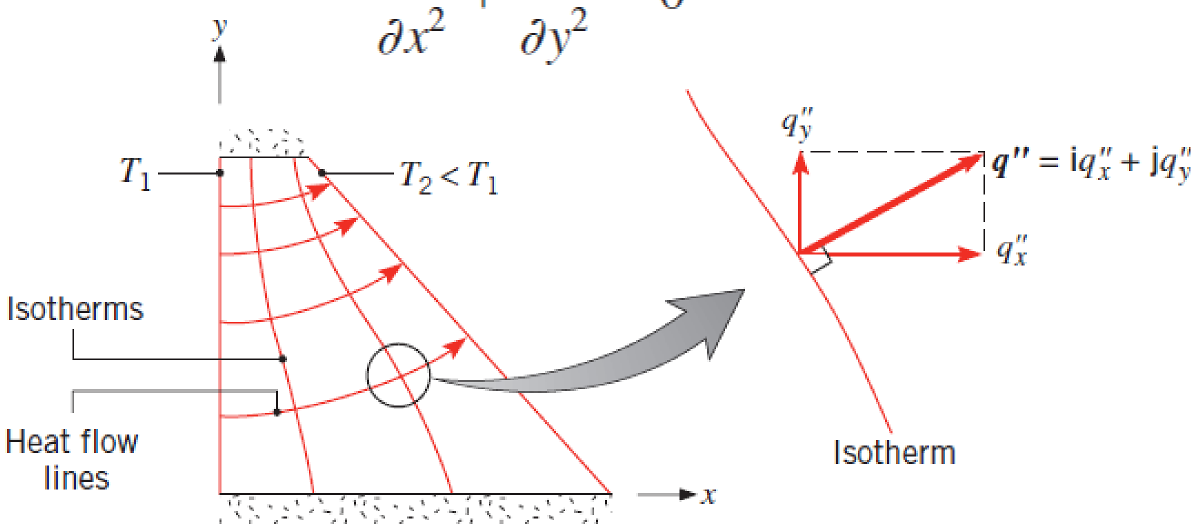


FIGURE 4.1 Two-dimensional conduction.

Two-Dimensional, Steady-State Conduction

1. Alternative Approaches

- The last equation can be solved using three various approaches giving the final equation to describe the temperature distribution along the two directions (x and y).

- Analytical: like variables separation and this can lead to an exact solution of the previous partial differential equation.

- Graphical: this method can provide only approximate results in a short time.

- Numerical (finite-difference, finite-element, or boundary-element): by this method, accurate results for complex, two- or three-dimensional geometries involving a wide variety of boundary conditions, can be reached.

Two-Dimensional, Steady-State Conduction

2. The Method of Separation of Variables

In order to obtain a simple solution; the transformation (θ) was introduced.

$$\theta \equiv \frac{T - T_1}{T_2 - T_1}$$

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

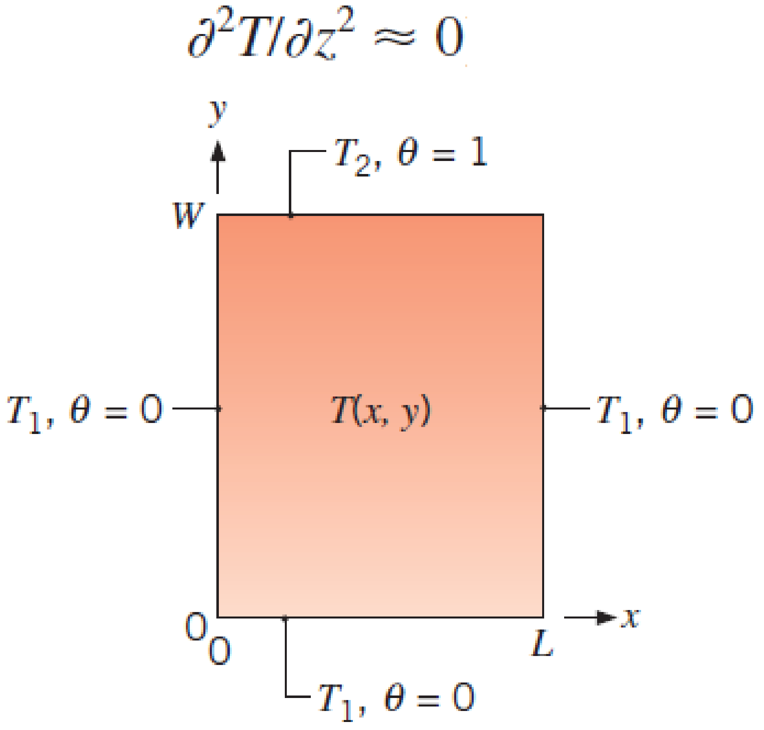


FIGURE 4.2 Two-dimensional conduction in a thin rectangular plate or a long rectangular rod.

Two-Dimensional, Steady-State Conduction

2. The Method of Separation of Variables

- Since the equation is second order in both x and y, two boundary conditions are needed for each of the coordinates.

$$\theta(0, y) = 0 \quad \text{and} \quad \theta(x, 0) = 0$$
$$\theta(L, y) = 0 \quad \text{and} \quad \theta(x, W) = 1$$

- Considering the solution is the product of two functions (one for and another for Y).

$$\theta(x, y) = X(x) \cdot Y(y)$$

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2}$$

$$\partial^2 T/\partial z^2 \approx 0$$

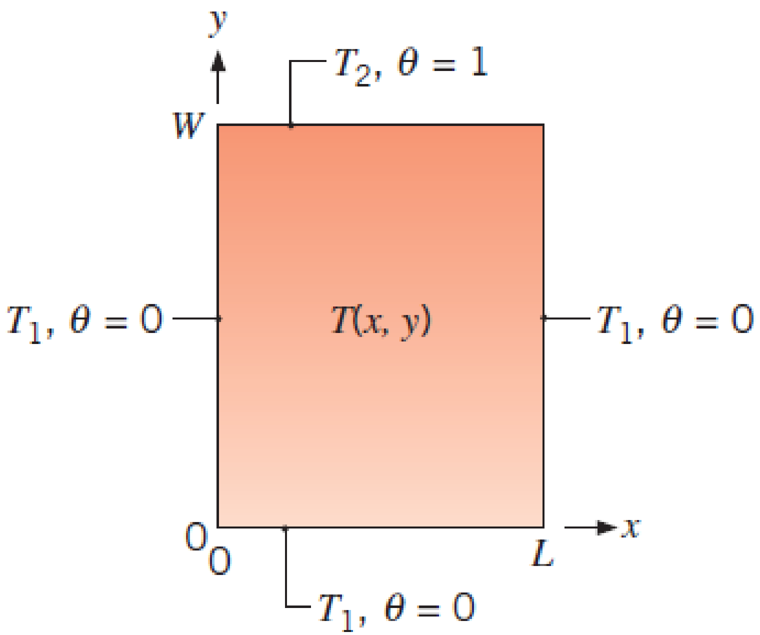


FIGURE 4.2 Two-dimensional conduction in a thin rectangular plate or a long rectangular rod.

Two-Dimensional, Steady-State Conduction

2. The Method of Separation of Variables

- Using the separation constant as λ^2:

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda^2$$

$$\boxed{\begin{aligned} &\frac{d^2X}{dx^2} + \lambda^2 X = 0 \\ &\frac{d^2Y}{dy^2} - \lambda^2 Y = 0 \end{aligned}}$$

- The last two equation are ordinary differential equations (one variable in each).

$\partial^2 T/\partial z^2 \approx 0$

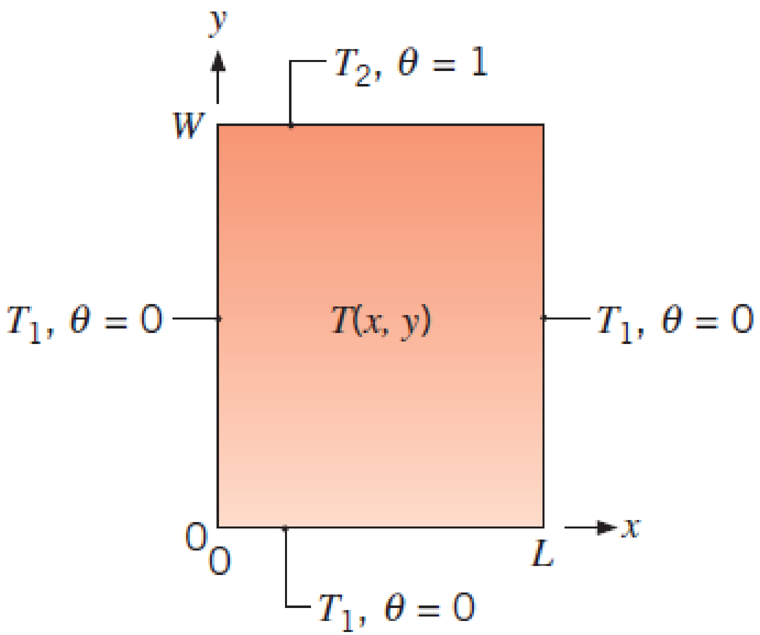


FIGURE 4.2 Two-dimensional conduction in a thin rectangular plate or a long rectangular rod.

Two-Dimensional, Steady-State Conduction

2. The Method of Separation of Variables

- The solution of the pervious equation is:

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$
$$Y = C_3 e^{-\lambda y} + C_4 e^{+\lambda y}$$

- Then the general solution is:

$$\theta = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda y} + C_4 e^{\lambda y})$$

- C1, C2, C3 and C4 are integration constant which can be determined using the boundary conditions (4 conditions) selected in the system.

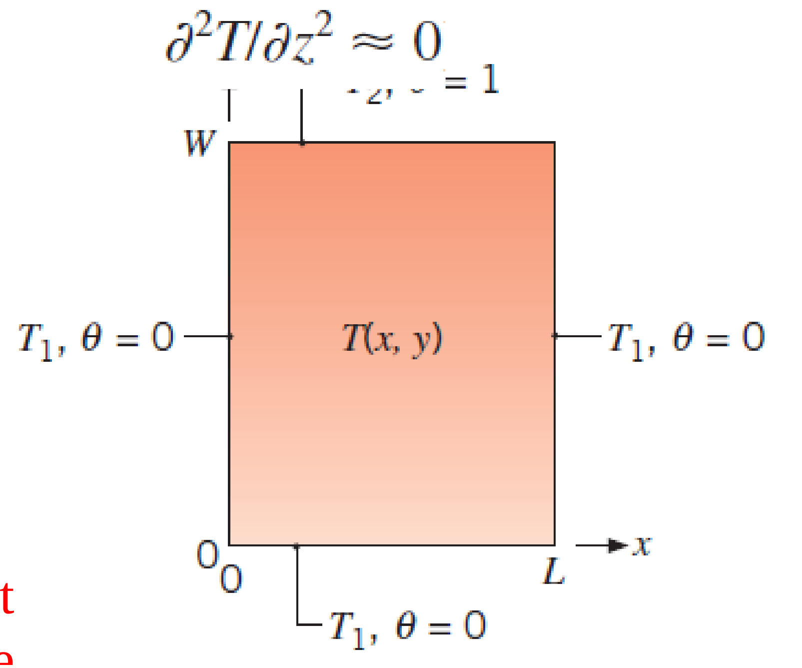


FIGURE 4.2 Two-dimensional conduction in a thin rectangular plate or a long rectangular rod.

Two-Dimensional, Steady-State Conduction

2. The Method of Separation of Variables

$$\theta = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda y} + C_4 e^{\lambda y})$$

1. x = 0 or y = 0:

$$\theta(0, y) = 0 \Rightarrow C_1 = 0$$

$$\theta(x, 0) = 0 \Rightarrow C_2 \sin \lambda x (C_3 + C_4) = 0$$

C_2= 0, this will result in:

$$\theta(x, y) = 0$$

and it does not satisfy the boundary condition;

$$\theta(x, W) = 1$$

The case, then, is that $C_3 + C_4 = 0$

Two-Dimensional, Steady-State Conduction

2. The Method of Separation of Variables

2. Using the condition in which: $\theta(L, y) = 0$

C_4 cannot be zero.

$$C_2C_4 \sin \lambda L(e^{\lambda y} - e^{-\lambda y}) = 0 \quad \Rightarrow \quad \sin \lambda L = 0$$

$$\lambda = \frac{n\pi}{L} \qquad n = 1, 2, 3, \ldots$$

n cannot be zero.

$$\theta = C_2C_4 \sin \frac{n\pi x}{L} (e^{n\pi y/L} - e^{-n\pi y/L})$$

$$\boxed{\theta(x, y) = C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}} \quad \Rightarrow \quad \boxed{\theta(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}}$$

Two-Dimensional, Steady-State Conduction

2. The Method of Separation of Variables

3. Using the condition in which: $\theta(x, W) = 1$

$$1 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi W}{L}$$

By applying a standard method, C_n can be determined finally using this equation:

$$C_n = \frac{2[(-1)^{n+1} + 1]}{n\pi \sinh (n\pi W/L)} \qquad n = 1, 2, 3, \ldots$$

$$\boxed{\theta(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin \frac{n\pi x}{L} \frac{\sinh (n\pi y/L)}{\sinh (n\pi W/L)}}$$

Two-Dimensional, Steady-State Conduction

2. The Method of Separation of Variables

$$\theta(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin \frac{n\pi x}{L} \frac{\sinh (n\pi y/L)}{\sinh (n\pi W/L)}$$

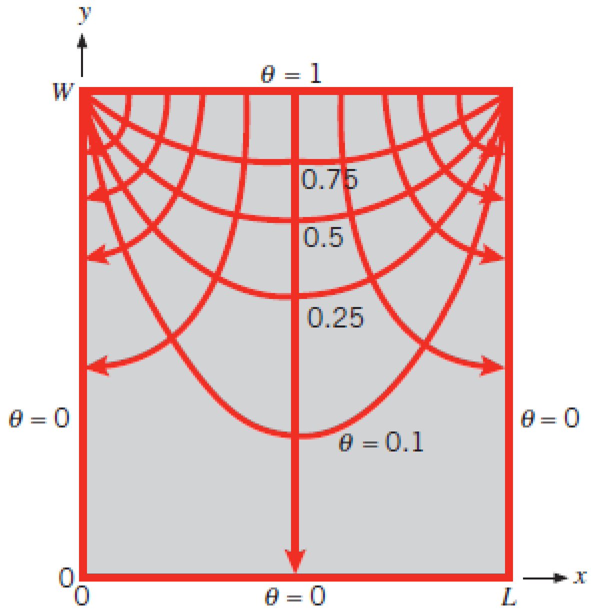


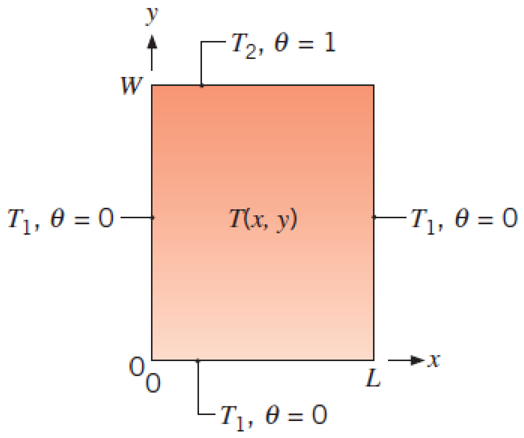
FIGURE 4.3 Isotherms and heat flow lines for two-dimensional conduction in a rectangular plate.

Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

- In general, finding analytical solutions to the two- or three-dimensional heat equation is time-consuming and, in many cases, not possible.
- Therefore, a different approach is often taken.
- This can be done using a shape factor S or a steady-state dimensionless conduction heat rate, q^*_{ss}:

$$q = Sk\Delta T_{1-2}$$

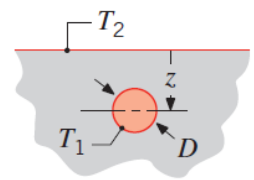
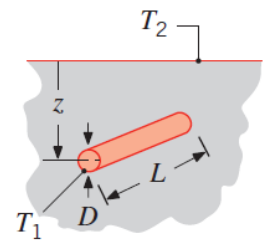


- Where $T_{1\text{-}2}$ is the temperature difference between boundaries.

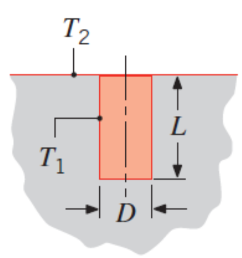
$$R_{t,cond(2D)} = \frac{1}{Sk}$$

Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

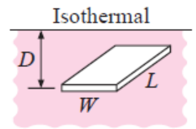
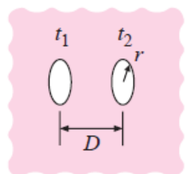
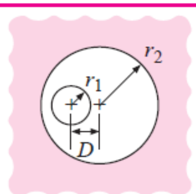
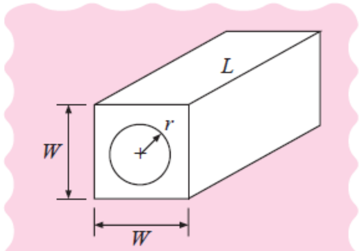
<table>
<tr><th>System</th><th>Schematic</th><th>Restrictions</th><th>Shape Factor</th></tr>
<tr><td>Case 1
Isothermal sphere buried in a semi-infinite medium</td><td>
</td><td>$z > D/2$</td><td>$\frac{2\pi D}{1 - D/4z}$</td></tr>
<tr><td>Case 2
Horizontal isothermal cylinder of length L buried in a semi-infinite medium</td><td>
</td><td>$L \gg D$

$L \gg D$
$z > 3D/2$</td><td>$\frac{2\pi L}{\cosh^{-1}(2z/D)}$

$\frac{2\pi L}{\ln(4z/D)}$</td></tr>
<tr><td>Case 3
Vertical cylinder in a semi-infinite medium</td><td>
</td><td>$L \gg D$</td><td>$\frac{2\pi L}{\ln(4L/D)}$</td></tr>
</table>

Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

Physical system	Schematic	Shape factor	Restrictions
Thin rectangular plate of length L, buried in semi-infinite medium having isothermal surface		$\dfrac{\pi W}{\ln(4W/L)}$ $\dfrac{2\pi W}{\ln(4W/L)}$ $\dfrac{2\pi W}{\ln(2\pi D/L)}$	$D = 0$, $W > L$ $D \gg W$, $W > L$ $W \gg L$, $D > 2W$
Parallel disks buried in infinite medium		$\dfrac{4\pi r}{\left[\frac{\pi}{2} - \tan^{-1}(r/D)\right]}$	$D > 5r$ $\tan^{-1}(r/D)$ in radians
Eccentric cylinders of length L		$\dfrac{2\pi L}{\cosh^{-1}\left(\dfrac{r_1^2+r_2^2-D^2}{2r_1r_2}\right)}$	$L \gg r_2$
Cylinder centered in a square of length L		$\dfrac{2\pi L}{\ln(0.54\,W/r)}$	$L \gg W$

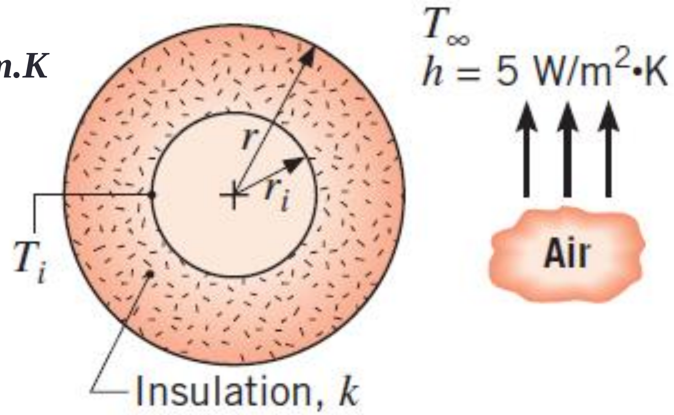
Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

Example-1:

The possible existence of an optimum insulation thickness for radial systems is suggested by the presence of competing effects associated with an increase in this thickness. In particular, although the conduction resistance increases with the addition of insulation, the convection resistance decreases due to increasing outer surface area. Hence there may exist an insulation thickness that minimizes heat loss by maximizing the total resistance to heat transfer. Resolve this issue by considering the following system.

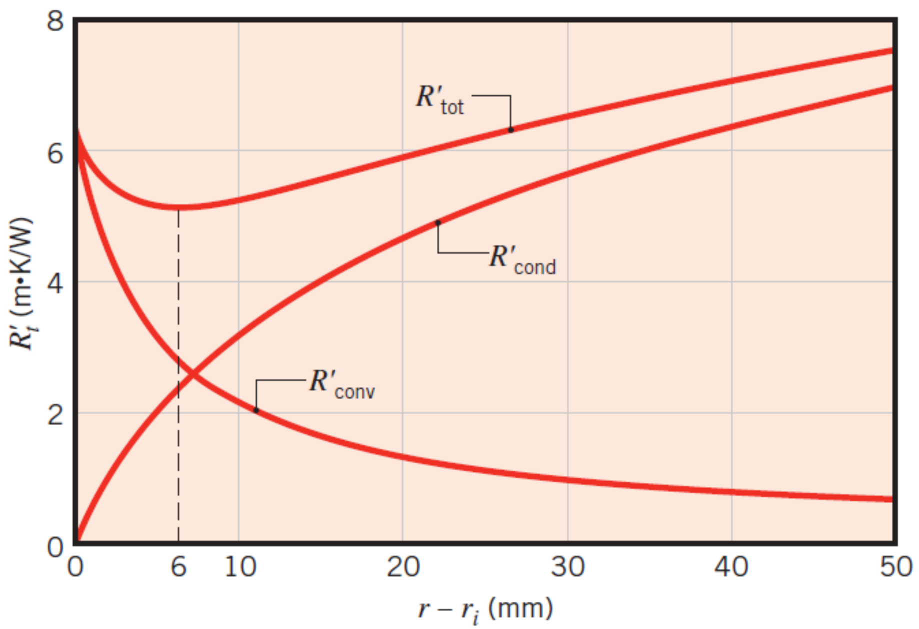
k=0.055 W/m.K



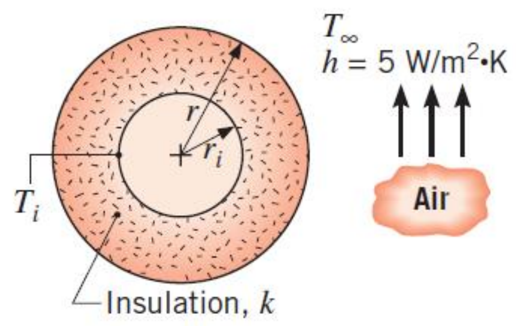
Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

Example-1:



k=0.055 W/m.K



Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

Example-1:

1. Steady-state conditions.

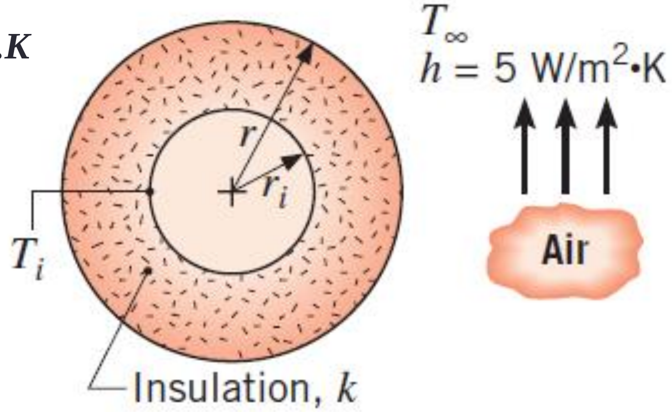
2. One-dimensional heat transfer in the radial (cylindrical) direction.

3. Negligible tube wall thermal resistance.

4. Constant properties for insulation.

5. Negligible radiation exchange between insulation outer surface and surroundings.

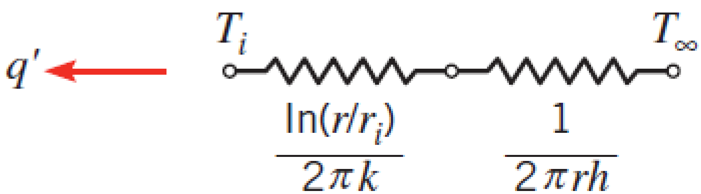
k=0.055 W/m.K

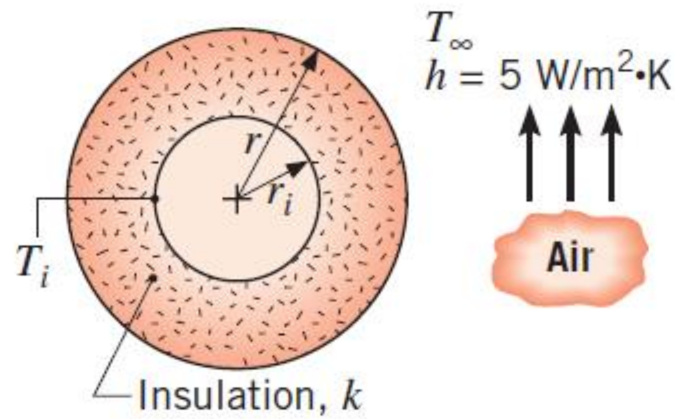


Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

Example-1:





$$R'_{\text{tot}} = \frac{\ln(r/r_i)}{2\pi k} + \frac{1}{2\pi rh}$$

$$q' = \frac{T_\infty - T_i}{R'_{\text{tot}}}$$

For optimum insulation thickness

$$\frac{dR'_{\text{tot}}}{dr} = 0 \quad \Rightarrow \quad \frac{1}{2\pi kr} - \frac{1}{2\pi r^2 h} = 0 \qquad \boxed{r = \frac{k}{h}}$$

Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

Example-1:

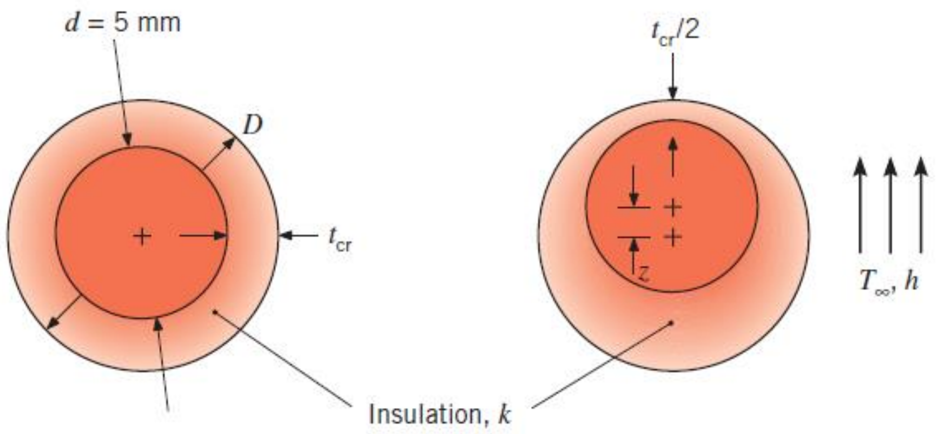
A metallic electrical wire of diameter d = 5 mm is to be coated with insulation of thermal conductivity k=0.35 W/m.K. It is expected that, for the typical installation, the coated wire will be exposed to conditions for which the total coefficient associated with convection and radiation is h = 15 W/m^2.K. To minimize the temperature rise of the wire due to ohmic heating, the insulation thickness is specified so that the *critical insulation radius* is achieved (as shown in the previous example). During the wire coating process, however, the insulation thickness sometimes varies around the wire. Determine the change in the thermal resistance of the insulation due to the variation that is 50% of the critical insulation thickness.

Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

Example-1:

1. Steady-state conditions.
2. Two-dimensional conduction.
3. Constant properties.
4. Both the exterior and interior surfaces of the coating are at uniform temperatures.

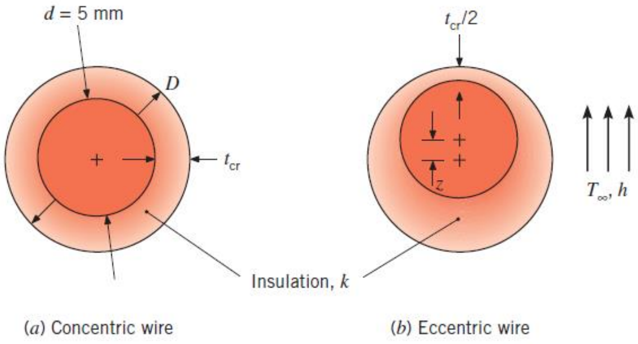


(*a*) Concentric wire (*b*) Eccentric wire

Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

Example-1:

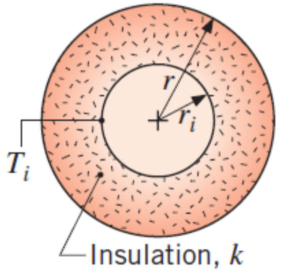


$$r_{cr} = \frac{k}{h} = \frac{0.35\ \text{W/m}\cdot\text{K}}{15\ \text{W/m}^2\cdot\text{K}} = 0.023\ \text{m} = 23\ \text{mm}$$

the critical insulation thickness is

$$t_{cr} = r_{cr} - d/2 = 0.023\ \text{m} - \frac{0.005\ \text{m}}{2} = 0.021\ \text{m} = 21\ \text{mm}$$

The thermal resistance of the coating associated with the concentric wire



$$R'_{t,\text{cond}} = \frac{\ln[r_{cr}/(d/2)]}{2\pi k} = \frac{\ln[0.023\ \text{m}/(0.005\ \text{m}/2)]}{2\pi(0.35\ \text{W/m}\cdot\text{K})} = 1.0\ \text{m}\cdot\text{K/W}$$

Two-Dimensional, Steady-State Conduction

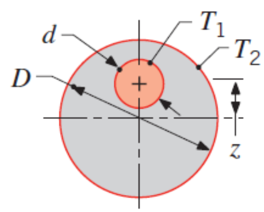
3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

Example-1:

For the eccentric wire, the thermal resistance of the insulation may be evaluated using

Case 7

Eccentric circular cylinder of length L in a cylinder of equal length



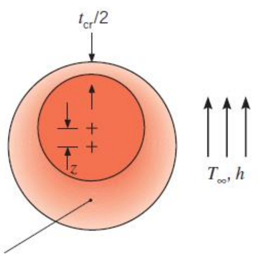
$D > d$
$L \gg D$

$$\frac{2\pi L}{\cosh^{-1}\left(\dfrac{D^2 + d^2 - 4z^2}{2Dd}\right)}$$

$$R'_{t,\text{cond(2D)}} = \frac{1}{Sk} = \frac{\cosh^{-1}\left(\dfrac{D^2 + d^2 - 4z^2}{2Dd}\right)}{2\pi k}$$

$$= \frac{\cosh^{-1}\left(\dfrac{(2 \times 0.023\text{ m})^2 + (0.005\text{ m})^2 - 4(0.010\text{ m})^2}{2 \times (2 \times 0.023\text{ m}) \times 0.005\text{ m}}\right)}{2\pi \times 0.35\text{ W/m}\cdot\text{K}}$$

$$= 0.91\text{ m}\cdot\text{K/W}$$



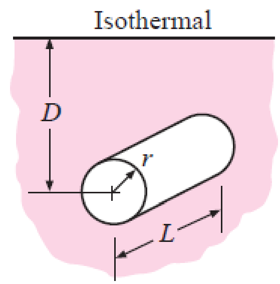
(*b*) Eccentric wire

Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

Example-2:

Horizontal pipe 15 cm in diameter and 4 m long is buried in the earth at a depth of 20 cm. The pipe-wall temperature is 75 °C, and the earth surface temperature is 5 °C. Assuming that the thermal conductivity of the earth is 0.8 W/m. °C, calculate the heat lost by the pipe.

Isothermal cylinder of radius r buried in semi-infinite medium having isothermal surface		$\frac{2\pi L}{\cosh^{-1}(D/r)}$	$L \gg r$
		$\frac{2\pi L}{\ln(D/r)}$	$L \gg r$ $D > 3r$

Since D<3r , We may calculate the shape factor for this situation using the equation given in in this table.

Two-Dimensional, Steady-State Conduction

3. The Conduction Shape Factor and the Dimensionless Conduction Heat Rate

Example-2:

$$S = \frac{2\pi L}{\cosh^{-1}(D/r)} = \frac{2\pi(4)}{\cosh^{-1}(20/7.5)} = 15.35 \text{ m}$$

The heat flow is calculated from

$$q = kS\Delta T = (0.8)(15.35)(75 - 5) = 859.6 \text{ W}$$