# Introduction to Robotics

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# Kinematic relations

$$coordinate - i : \begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$$

Joint coordinate-i: $\boxed{q_i = \bar{\varepsilon}_i \theta_i + \varepsilon_i d_i}$

with $\varepsilon_i = \begin{cases} 0 & \text{revolute} \\ 1 & \text{prismatic} \end{cases}$

and $\bar{\varepsilon}_i = 1 - \varepsilon_i$

Joint Coordinate Vector: $\boxed{q = (q_1 q_2 \ldots . q_n)^T}$

X=FK(q)

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \varphi \end{bmatrix}$$

q =IK(X)

Joint Space

Task Space

Location of the tool can be specified using a joint space or a cartesian space description

# Velocity relations

- Relation between joint velocity and Cartesian velocity.
- JACOBIAN matrix J(q)

$$coordinate - i : \begin{cases} \theta_i & \text{revolute} \\ \mathrm{d}_i & \text{prismatic} \end{cases}$$

Joint coordinate-i: $\boxed{q_i = \bar{\varepsilon}_i \theta_i + \varepsilon_i d_i}$

with $\varepsilon_i = \begin{cases} 0 & \text{revolute} \\ 1 & \text{prismatic} \end{cases}$

and $\bar{\varepsilon}_i = 1 - \varepsilon_i$

Joint Coordinate Vector: $\boxed{q = (q_1 q_2 \dots q_n)^T}$

$$q^{\cdot} = \begin{bmatrix} q_1^{\cdot} \\ q_2^{\cdot} \\ q_3^{\cdot} \\ q_4^{\cdot} \\ q_5^{\cdot} \\ q_6^{\cdot} \end{bmatrix} \underset{\dot{q} = J^{-1}(q) X^{\cdot}}{\overset{\dot{X} = J(q) q^{\cdot}}{\rightleftarrows}} X^{\cdot} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \phi^{\cdot} \\ \theta^{\cdot} \\ \varphi^{\cdot} \end{bmatrix}$$

Joint Space

Task Space

# Jacobian

- Suppose a position and orientation vector of a manipulator is a function of 6 joint variables: (from forward kinematics)

$$X = h(q)$$

$$X = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \varphi \end{bmatrix} = h\left(\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}\right)_{6\times 1} = \begin{bmatrix} h_1(q_1, q_2, \cdots, q_6) \\ h_2(q_1, q_2, \cdots, q_6) \\ h_3(q_1, q_2, \cdots, q_6) \\ h_4(q_1, q_2, \cdots, q_6) \\ h_5(q_1, q_2, \cdots, q_6) \\ h_6(q_1, q_2, \cdots, q_6) \end{bmatrix}_{6\times 1}$$

$$coordinate-i: \begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$$

Joint coordinate-i: $\boxed{q_i = \bar{\varepsilon}_i \theta_i + \varepsilon_i d_i}$

with $\varepsilon_i = \begin{cases} 0 & \text{revolute} \\ 1 & \text{prismatic} \end{cases}$

and $\bar{\varepsilon}_i = 1 - \varepsilon_i$

Joint Coordinate Vector: $\boxed{q = (q_1 q_2 \dots q_n)^T}$

# Jacobian Matrix

Forward kinematics

$coordinate - i : \begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$

Joint coordinate-i: $\boxed{q_i = \bar{\varepsilon}_i \theta_i + \varepsilon_i d_i}$

with $\varepsilon_i = \begin{cases} 0 & \text{revolute} \\ 1 & \text{prismatic} \end{cases}$

and $\bar{\varepsilon}_i = 1 - \varepsilon_i$

Joint Coordinate Vector: $\boxed{q = (q_1 q_2 \ldots q_n)^T}$

$$X_{6\times1} = h(q_{n\times1})$$

$$\dot{X}_{6\times1} = \frac{d}{dt} h(q_{n\times1}) = \frac{d}{dq} h(q) \cdot \frac{dq}{dt} = \frac{d}{dt} h(q) \dot{q}$$

$$\dot{X} = J_{6\times n} \, \dot{q}_{n\times1}$$

$$\boxed{J = \frac{dh(q)}{dq}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \boxed{\left[\frac{d}{dq} h(q)\right]} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times1}$$

# Jacobian Matrix

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \left[\frac{d}{dq} h(q)\right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

Jacobian is a function of $q$, it is not a constant!

$$J = \left(\frac{dh(q)}{dq}\right)_{6\times n} = \begin{bmatrix} \frac{\partial h_1}{\partial q_1} & \frac{\partial h_1}{\partial q_2} & \cdots & \frac{\partial h_1}{\partial q_n} \\ \frac{\partial h_2}{\partial q_1} & \frac{\partial h_2}{\partial q_1} & \cdots & \frac{\partial h_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_6}{\partial q_1} & \frac{\partial h_6}{\partial q_1} & \cdots & \frac{\partial h_6}{\partial q_n} \end{bmatrix}_{6\times n}$$

# Jacobian Matrix

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} V \\ \Omega \end{bmatrix}$$

Linear velocity

$$V = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

Angular velocity

$$\Omega = \begin{bmatrix} \omega_x = \dot{\phi} \\ \omega_y = \dot{\theta} \\ \omega_z = \dot{\psi} \end{bmatrix}$$

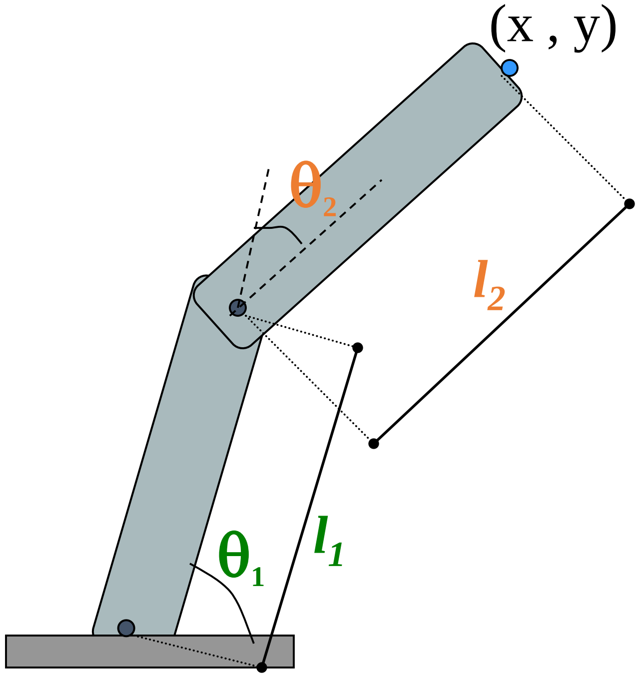
$$\dot{q}_{n\times 1} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

**The Jacbian Equation**

$$\dot{X} = J_{6\times n}\dot{q}_{n\times 1}$$

# Example

- 2-DOF planar robot arm
  - Given ***$l_1$, $l_2$, Find: Jacobian***




$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) \end{bmatrix} = \begin{bmatrix} h_1(\theta_1,\theta_2) \\ h_2(\theta_1,\theta_2) \end{bmatrix}$$

$$\dot{Y} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$J = \begin{bmatrix} \dfrac{\partial h_1}{\partial \theta_1} & \dfrac{\partial h_1}{\partial \theta_2} \\ \dfrac{\partial h_2}{\partial \theta_1} & \dfrac{\partial h_2}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix}$$

# Singularities

- The inverse of the jacobian matrix cannot be calculated when

$$\det[J(\theta)] = 0$$

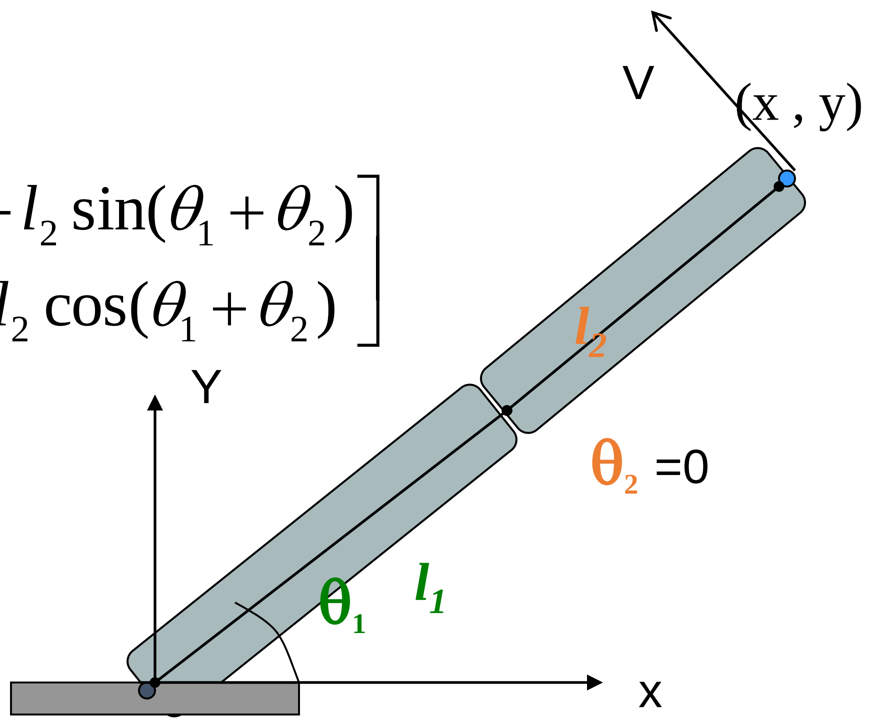
- Singular points are such values of $\theta$ that cause the determinant of the Jacobian to be zero

- Find the singularity configuration of the 2-DOF planar robot arm

determinant(J)=0 → Not full rank

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$J = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$




Det(J)=0

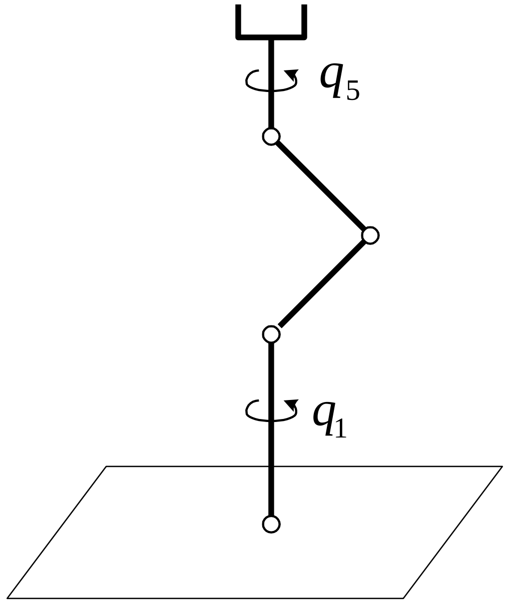
$$\theta_2 = 0$$

# Jacobian Matrix

- Inverse Jacobian

$$\dot{X} = J\dot{q} = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{16} \\ J_{21} & J_{22} & \cdots & J_{26} \\ \vdots & \vdots & \vdots & \vdots \\ J_{61} & J_{62} & \cdots & J_{66} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

$$\dot{q} = J^{-1}\dot{X}$$




- Singularity
  - rank(J)<n : Jacobian Matrix is less than full rank
  - Jacobian is non-invertable
  - Boundary Singularities: occur when the tool tip is on the surface of the work envelop.
  - Interior Singularities: occur inside the work envelope when two or more of the axes of the robot form a straight line, i.e., collinear

# Singularity

- At Singularities:
  - the manipulator end effector cant move in certain directions.
  - Bounded End-Effector velocities may correspond to unbounded joint velocities.
  - Bounded joint torques may correspond to unbounded End-Effector forces and torques.

# Jacobian Matrix

- If $A = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, B = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$

- Then the cross product

$$A \times B = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ -(a_x b_z - a_z b_x) \\ a_x b_y - a_y b_x \end{bmatrix}$$

# Remember DH parmeter

$$
{}^{i-1}_{i}H = \begin{bmatrix} \mathrm{c}\theta_{\mathrm{i}} & -\mathrm{c}\alpha_{\mathrm{i}}\mathrm{s}\theta_{\mathrm{i}} & \mathrm{s}\alpha_{\mathrm{i}}\mathrm{s}\theta_{\mathrm{i}} & \mathrm{a}_{\mathrm{i}}\mathrm{c}\theta_{\mathrm{i}} \\ \mathrm{s}\theta_{\mathrm{i}} & \mathrm{c}\theta_{\mathrm{i}}\mathrm{c}\alpha_{\mathrm{i}} & -\mathrm{s}\alpha_{\mathrm{i}}\mathrm{c}\theta_{\mathrm{i}} & \mathrm{a}_{\mathrm{i}}\mathrm{s}\theta_{\mathrm{i}} \\ 0 & \mathrm{s}\alpha_{\mathrm{i}} & \mathrm{c}\alpha_{\mathrm{i}} & \mathrm{d}_{\mathrm{i}} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- The transformation matrix H

$$
{}^{0}_{6}H(q_1,\ldots,q_n) = {}^{0}_{1}H(q_1)\cdots{}^{n-1}_{n}H(q_n)
$$

# Jacobian Matrix

$$J = \begin{bmatrix} J_1 & J_2 & \dots & J_n \end{bmatrix}$$

$${}^{i-1}_{\ \ i}H = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where if joint *(i)* is revolute

$$J_i = \begin{bmatrix} Z_{i-1} \times (O_n - O_{i-1}) \\ Z_{i-1} \end{bmatrix}$$

and if joint *(i)* is prismatic

$$J_i = \begin{bmatrix} Z_{i-1} \\ 0 \end{bmatrix}$$

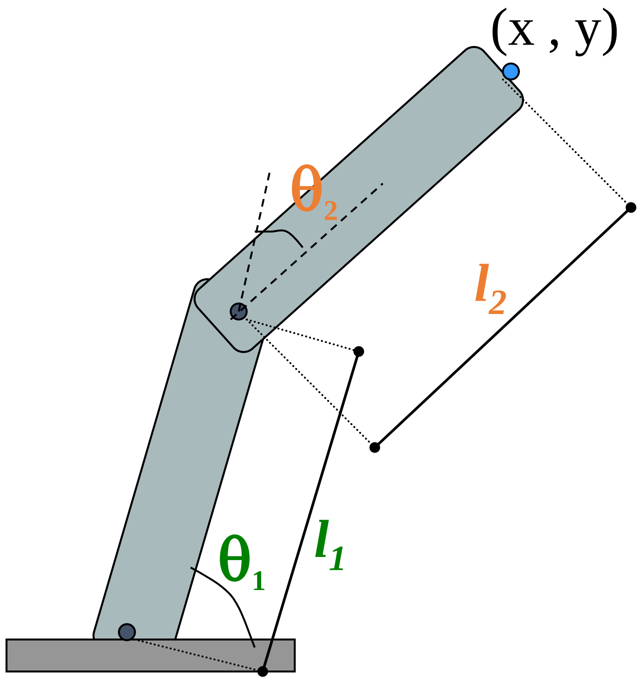
Where $Z_i$ is the first three elements in the 3rd column of the ${}^{i-1}_{\ \ i}H$ matrix, and $O_i$ is the first three elements in the 4th column of the ${}^{i-1}_{\ \ i}H$ matrix.

# Jacobian Matrix

2-DOF planar robot arm

Given $l_1$, $l_2$ , ***Find: Jacobian***

- ***Here, n=2,***



| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $a_1$ | 0 | 0 | $\theta_1^*$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2^*$ |

* variable

$$
{}^0_1H = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

$$
{}^1_2H = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Where $(\theta_1 + \theta_2)$ denoted by $\theta_{12}$ and $\cos(\theta_1 + \theta_2)$ by $c_{12}$

$$
{}^0_1H = {}^0_1H
$$

$$
Z_0 = Z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
{}^0_2H = {}^0_1H\,{}^1_2H = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
O_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, O_1 = \begin{bmatrix} a_1\cos\theta_1 \\ a_1\sin\theta_1 \\ 0 \end{bmatrix}, O_2 = \begin{bmatrix} a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) \\ a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) \\ 0 \end{bmatrix}
$$

# Jacobian Matrix

2-DOF planar robot arm

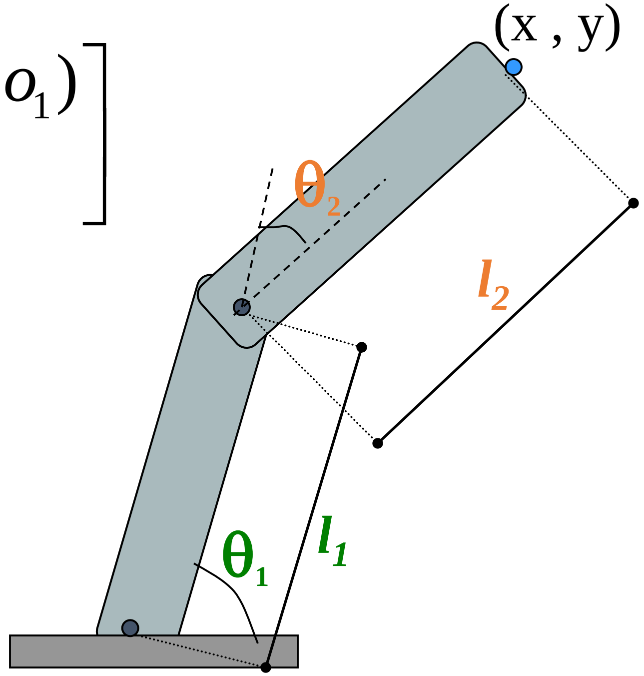
Given ***$l_1$, $l_2$* , *Find: Jacobian***

| Link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $a_1$ | 0 | 0 | $\theta_1^*$ |
| 2 | $a_2$ | 0 | 0 | $\theta_2^*$ |

* variable

- ***Here, n=2***

$$J_1 = \begin{bmatrix} z_0 \times (o_2 - o_0) \\ z_0 \end{bmatrix}, J_2 = \begin{bmatrix} z_1 \times (o_2 - o_1) \\ z_1 \end{bmatrix}$$



# Jacobian Matrix

$$J_1 = \begin{bmatrix} z_0 \times (o_2 - o_0) \\ z_0 \end{bmatrix}$$

$$Z_0 \times (o_2 - o_0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

# Jacobian Matrix

$$J_2 = \begin{bmatrix} z_1 \times (o_2 - o_1) \\ z_1 \end{bmatrix}$$

$$Z_1 \times (o_2 - o_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 \cos(\theta_1 + \theta_2) \\ a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ a_2 \cos(\theta_1 + \theta_2) & a_2 \sin(\theta_1 + \theta_2) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_2 \sin(\theta_1 + \theta_2) \\ a_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

# Jacobian Matrix

$$J_1 = \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad J_2 = \begin{bmatrix} -a_2 \sin(\theta_1 + \theta_2) \\ a_2 \cos(\theta_1 + \theta_2) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

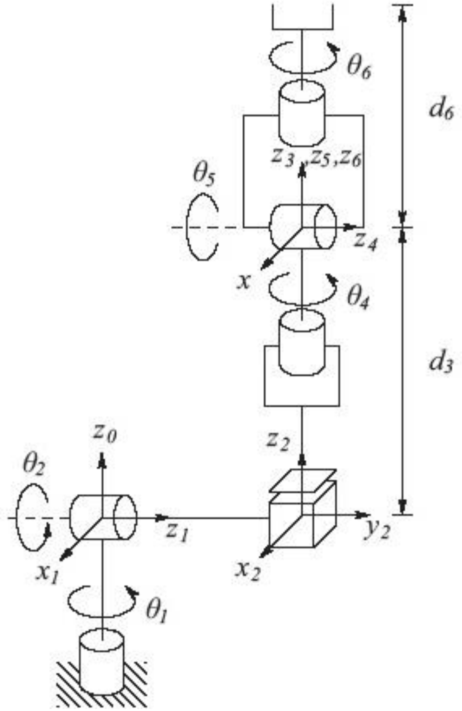
The required Jacobian matrix **J**

$$J = \begin{bmatrix} J_1 & J_2 \end{bmatrix}$$

# Stanford Manipulator

- 6DOF: need to assign seven coordinate frames:
  1. Choose $z_0$ axis (axis of rotation for joint 1, base frame)
  2. Choose $z_1$-$z_5$ axes (axes of rotation/translation for joints 2-6)
  3. Choose $x_i$ axes
  4. Choose tool frame
  5. Fill in table of DH parameters:

| link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | -90 | 0 | $\theta_1$ |
| 2 | 0 | 90 | $d_2$ | $\theta_2$ |
| 3 | 0 | 0 | $d_3$ | 0 |
| 4 | 0 | -90 | 0 | $\theta_4$ |
| 5 | 0 | 90 | 0 | $\theta_5$ |
| 6 | 0 | 0 | $d_6$ | $\theta_6$ |



- Now determine the individual homogeneous transformations:

$$
{}^0_1H = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^1_2H = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^2_3H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^3_4H = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^4_5H = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^5_6H = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

# Stanford Manipulator

$$ {}^0_1H = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1_2H = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^2_3H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_2H = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & -d_2s_1 \\ s_1c_2 & c_1 & s_1s_2 & d_2c_1 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} z_2 = \begin{bmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \end{bmatrix} z_3 = \begin{bmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \end{bmatrix} $$

$$ {}^0_3H = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & d_3c_1s_2 - d_2s_1 \\ s_1c_2 & c_1 & s_1s_2 & d_3s_1s_2 + d_2c_1 \\ -s_2 & 0 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ O_0 = O_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} O_2 = \begin{bmatrix} -d_2s_1 \\ d_2c_1 \\ 0 \end{bmatrix} O_3 = \begin{bmatrix} d_3c_1s_2 - d_2s_1 \\ d_3s_1s_2 + d_2c_1 \\ d_3c_2 \end{bmatrix} $$

# Stanford Manipulator

$$z_4 = \begin{bmatrix} -c_1c_2s_4 - s_1c_4 \\ -s_1c_2s_4 + c_1c_4 \\ s_2s_4 \end{bmatrix} \qquad O_4 = \begin{bmatrix} d_3c_1s_2 - d_2s_1 \\ d_3s_1s_2 + d_2c_1 \\ d_3c_2 \end{bmatrix}$$

$$z_5 = \begin{bmatrix} c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2c_5 \\ s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 \\ -s_2c_4s_5 + c_2c_5 \end{bmatrix} \qquad O_5 = \begin{bmatrix} d_3c_1s_2 - d_2s_1 \\ d_3s_1s_2 + d_2c_1 \\ d_3c_2 \end{bmatrix}$$

$$z_6 = ? \qquad O_6 = ?$$

# Stanford Manipulator

$$J_1 = \begin{bmatrix} z_0 \times (o_6 - o_0) \\ z_0 \end{bmatrix}, J_2 = \begin{bmatrix} z_1 \times (o_6 - o_1) \\ z_1 \end{bmatrix}$$

Joints 1,2 are revolute

$$J_3 = \begin{bmatrix} z_2 \\ 0 \end{bmatrix}$$

Joint 3 is prismatic

$$J_4 = \begin{bmatrix} z_3 \times (o_6 - o_3) \\ z_3 \end{bmatrix}, J_5 = \begin{bmatrix} z_4 \times (o_6 - o_4) \\ z_4 \end{bmatrix}, J_6 = \begin{bmatrix} z_5 \times (o_6 - o_5) \\ z_5 \end{bmatrix}$$

The required Jacobian matrix **J**

$$J = \begin{bmatrix} J_1 & J_2 J_3 J_4 J_5 J_6 \end{bmatrix}$$

$$J_1 = \begin{bmatrix} -d_y \\ d_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, J_2 = \begin{bmatrix} c_1 d_z \\ s_1 d_z \\ -s_1 d_y - c_1 d_x \\ -s_1 \\ c_1 \\ 0 \end{bmatrix}, J_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, J_4 = \begin{bmatrix} s_1 s_2 \left(d_z - o_{3,z}\right) + c_2 \left(d_y - o_{3,y}\right) \\ -c_1 s_1 \left(d_z - o_{3,z}\right) + c_2 \left(d_x - o_{3,x}\right) \\ -c_1 c_2 s_4 - s_1 c_4 \\ s_2 s_4 \\ 0 \\ 0 \end{bmatrix}$$

$$J_5 = \begin{bmatrix} \left(-s_1 c_2 s_4 + c_1 c_4\right)\left(d_z - o_{3,z}\right) - s_2 s_4 \left(d_y - o_{3,y}\right) \\ \left(-c_1 c_2 s_4 + s_1 c_4\right)\left(d_z - o_{3,z}\right) + s_2 s_4 \left(d_x - o_{3,x}\right) \\ \left(-c_1 c_2 s_4 - s_1 c_4\right)\left(d_y - o_{3,y}\right) + \left(s_1 c_2 s_4 - c_1 c_4\right)\left(d_x - o_{3,x}\right) \\ -c_1 c_2 c_4 - s_1 c_4 \\ s_2 s_4 \\ 0 \end{bmatrix}$$

$$J_6 = \begin{bmatrix} \left(s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5\right)\left(d_y - o_{3,y}\right) + \left(s_2 c_4 s_5 - c_2 c_5\right)\left(d_y - o_{3,y}\right) \\ -\left(c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5\right)\left(d_z - o_{3,z}\right) + \left(s_2 c_4 s_5 - c_2 c_5\right)\left(d_x - o_{3,x}\right) \\ c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c \\ -s_2 c_4 s_5 + c_2 c_5 \\ 0 \end{bmatrix}$$

# Inverse Velocity

- The relation between the joint and end-effector velocities:

$$\dot{X} = J(q)\dot{q}$$

where j (m×n). If J is a square matrix (m=n), the joint velocities:

$$\dot{q} = J^{-1}(q)\dot{X}$$

- If m<n, let pseudoinverse $J^+$ where

$$\dot{q} = J^{+}(q)\dot{X} \qquad J^{+} = J^{T}[JJ^{T}]^{-1}$$

# Jacobian Matrix

- Pseudoinverse
  - Let A be an mxn matrix, and let $A^+$ be the pseudoinverse of A. If A is of full rank, then $A^+$ can be computed as:

$$A^+ = \begin{cases} A^T[AA^T]^{-1} & m \le n \\ A^{-1} & m = n \\ [A^TA]^{-1}A^T & m \ge n \end{cases}$$

# Jacobian Matrix

- Example: Find X s.t.

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$A^{+} = A^{T}[AA^{T}]^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{9} \begin{bmatrix} 1 & 4 \\ 1 & -5 \\ 4 & -2 \end{bmatrix}$$

Matlab Command: [pinv(A)]() to calculate $A^{+}$

$$x = A^{+}b = \frac{1}{9} \begin{bmatrix} -5 \\ 13 \\ 16 \end{bmatrix}$$

# Acceleration

- The relation between the joint and end-effector velocities:

$$\dot{X} = J(q)\dot{q}$$

- Differentiating this equation yields an expression for the acceleration:

$$\ddot{X} = J(q)\ddot{q} + [\frac{d}{dt} J(q)]\dot{q}$$

- Given $\ddot{X}$ of the end-effector acceleration, the joint acceleration $\ddot{q}$

$$J(q)\ddot{q} = \ddot{X} - [\frac{d}{dt} J(q)]\dot{q}$$

$$\ddot{q} = J^{-1}(q)[\ddot{X} - \frac{d}{dt} J(q)]\dot{q}]$$

# Inverse kinematics algorithms

The Jacobian matrix can be used also for the solution of the inverse kinematic problem.

If a desired trajectory is known in terms of the velocity $\mathbf{v}(kT) = \mathbf{v}_k$, a simple approach is to consider

$$\mathbf{q}_{k+1} = \mathbf{q}_k + \mathbf{J}^{-1}(\mathbf{q}_k)\mathbf{v}_k T$$

equivalent to a numerical integration over time of the velocity. This operation has two major drawbacks affecting the computation of the exact solution:

- *numerical drifts*
- *initialization problems*

These problems may be avoided by implementing a feedback scheme accounting for the operational space errors $\mathbf{e} = \mathbf{x}_d - \mathbf{x}$. Then

$$\dot{\mathbf{e}} = \dot{\mathbf{x}}_d - \dot{\mathbf{x}} = \dot{\mathbf{x}}_d - \mathbf{J}_a(\mathbf{q})\dot{\mathbf{q}}$$

The vector $\dot{\mathbf{q}}$ must be chosen so that the error $\mathbf{e}$ converges to zero.

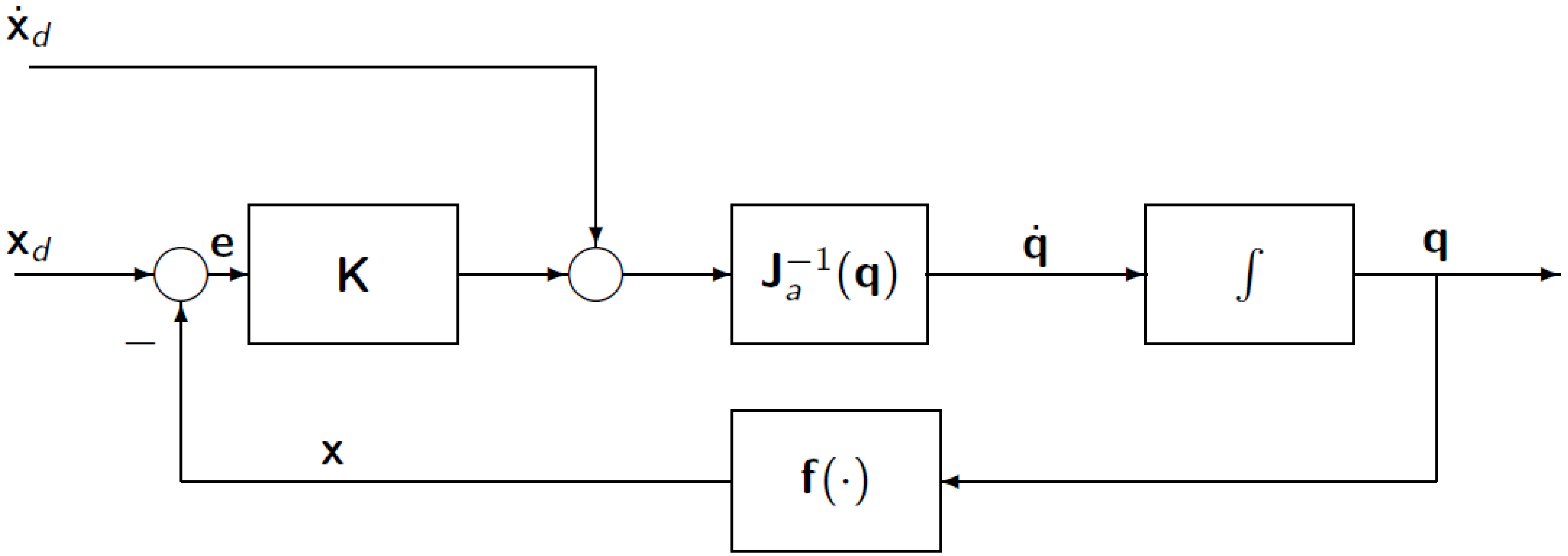
# Scheme 1: Jacobian (pseudo)-inverse

Assuming that $\mathbf{J}_a$ is invertible, then $\dot{\mathbf{q}} = \mathbf{J}_a^{-1}(\mathbf{q})\,(\dot{\mathbf{x}}_d + \mathbf{K}\mathbf{e})$

By substituting this expression in (2) one obtains

$$\dot{\mathbf{e}} + \mathbf{K}\mathbf{e} = 0$$

representing, if $\mathbf{K}$ is positive definite, an asymptotically stable system. Note that the convergence velocity depends on $\mathbf{K}$.



## Scheme 2: Jacobian transpose

This scheme is based on the Lyapunov approach. A Lyapunov function must be found guaranteeing the convergence to zero of the error $\mathbf{e}$.

Let's assume

$$V(\mathbf{e}) = \frac{1}{2}\mathbf{e}^T\ \mathbf{K}\ \mathbf{e}$$

with $\mathbf{K}$ symmetric and positive definite. In this manner

$$V(\mathbf{e}) > 0, \quad \forall \mathbf{e} \neq 0, \qquad V(\mathbf{0}) = 0$$

By differentiating $V(\mathbf{e})$ one obtains

$$\begin{aligned} \dot{V} &= \mathbf{e}^T\mathbf{K}\dot{\mathbf{e}} = \mathbf{e}^T\mathbf{K}\dot{\mathbf{x}}_d - \mathbf{e}^T\mathbf{K}\dot{\mathbf{x}} \\ &= \mathbf{e}^T\mathbf{K}\dot{\mathbf{x}}_d - \mathbf{e}^T\mathbf{K}\mathbf{J}_a(\mathbf{q})\dot{\mathbf{q}} \end{aligned}$$

# Scheme 2: Jacobian transpose

From

$$\dot{V} = \mathbf{e}^T \mathbf{K} \dot{\mathbf{x}}_d - \mathbf{e}^T \mathbf{K} \mathbf{J}_a(\mathbf{q}) \dot{\mathbf{q}}$$

By choosing

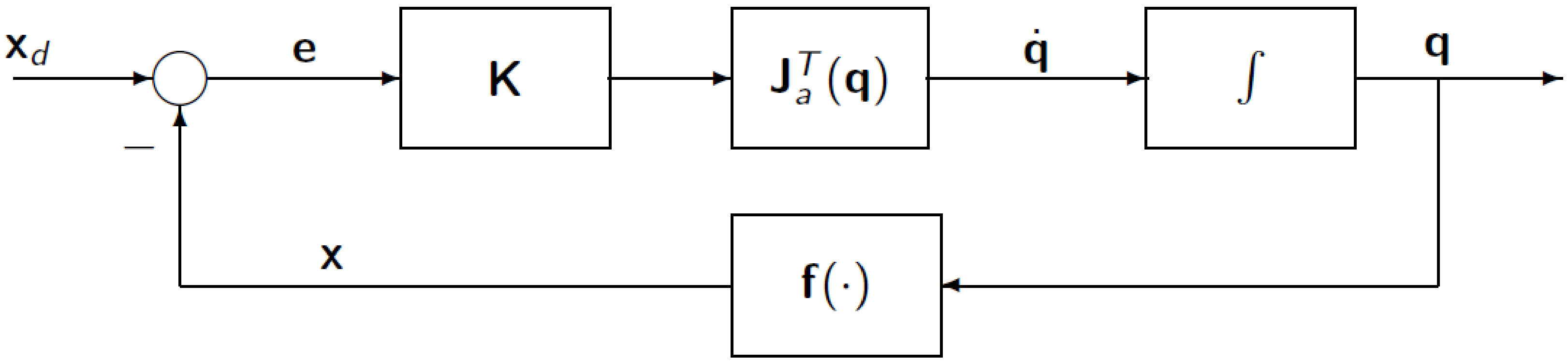
$$\dot{\mathbf{q}} = \mathbf{J}_a^T(\mathbf{q}) \mathbf{K} \mathbf{e}$$

one obtains

$$\dot{V} = \mathbf{e}^T \mathbf{K} \dot{\mathbf{x}}_d - \mathbf{e}^T \mathbf{K} \mathbf{J}_a(\mathbf{q}) \mathbf{J}_a^T(\mathbf{q}) \mathbf{K} \mathbf{e}$$

If $\dot{\mathbf{x}}_d = 0$ then $\dot{V} < 0$ and the system is asymptotically stable if $\mathbf{J}_a$ is full rank.

If $\mathbf{J}_a$ is not full rank, then the condition $\dot{\mathbf{q}} = 0$ may be obtained also with $\mathbf{e} \neq \mathbf{0}$ ($\mathbf{K}\mathbf{e} \in null(\mathbf{J}_a^T)$).



## Example

Consider the non linear function

$$\mathbf{z} = \mathbf{f}(\mathbf{q}) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} q_1^3 + sin(q_1 q_2) \\ q_1 q_2^3 + sin(q_1^2 + 2q_2) \end{bmatrix}$$

The Jacobian is

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} 3q_1^2 + q_2 cos(q_1 q_2) & q_1 cos(q_1 q_2) \\ q_2^3 + 2q_1 cos(q_1^2 + 2q_2) & 3q_1 q_2^2 + 2cos(q_1^2 + 2q_2) \end{bmatrix}$$

Assuming $\mathbf{z}_0 = [0.1,\ 0.2]^T$, find $\mathbf{q}_0 = \mathbf{f}^{-1}(\mathbf{z}_0)$.