ELECTRODYNAMICS

- We now want to consider time dependent $\vec{E}$ and $\vec{B}$ fields, as well as sources & sinks for currents.

Electromotive force

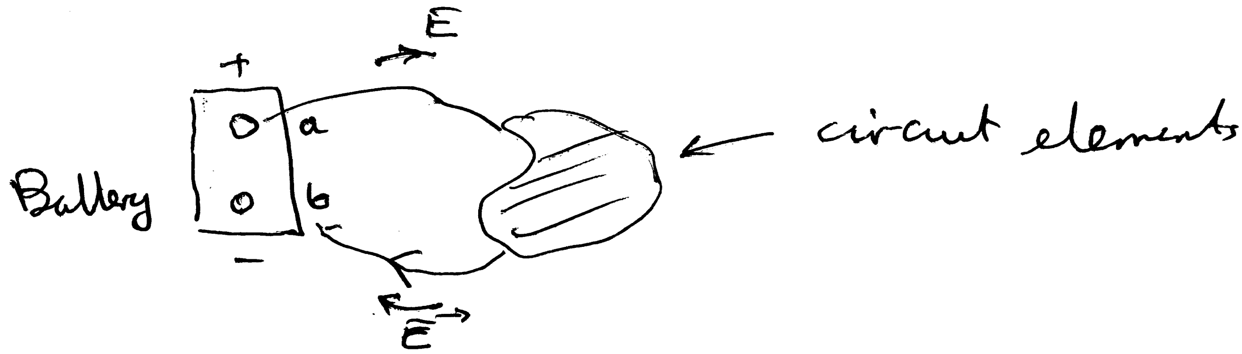
- To make a current flow in a circuit, there has to be a force to drive the electric charges around the circuit.

- If $f(\vec{r}) = \frac{F(\vec{r})}{q}$ is the force per unit charge at a point in the circuit, the <u>electromotive force</u> (emf, or $\mathcal{E}$) is defined as the <u>work per unit charge</u> in moving the charge around the circuit:

$$\mathcal{E} = \frac{1}{q} \oint_{T} \vec{F}(\vec{r}) \cdot d\vec{l}$$

Note: emf is NOT A FORCE, IT IS THE RESULT OF A FORCE.

- For an electric circuit joined to a battery or generator: an electric potential difference ΔV is set up between the ends of the circuit $\Rightarrow$ there is an electric field in the circuit $\Rightarrow$ a force $q\vec{E}(\vec{r})$ on charges in the wire.



$$\mathcal{E} = \frac{1}{q}\int_a^b \vec{F}(\vec{r}).\vec{dl}$$

$$= \frac{1}{q}\int_a^b q\vec{E}(\vec{r}).\vec{dl}$$

$$= \int_a^b \vec{E}(\vec{r}).\vec{dl}$$

$$= \int_a^b (-\vec{\nabla} V(\vec{r})).\vec{dl}$$

$$= -(V(b) - V(a))$$

$$= V(a) - V(b)$$

difference in potential across the terminals of the battery, unit <u>VOLTS</u>.

<u>NOTE</u>: in electrostatics, we said there can be no $\vec{E}$ field in a conductor. The above has <u>moving</u> charges, $\vec{E}$ fields exist in the conductor.

So: work $q(V(a) - V(b))$ is done in moving the charge around the conductor $\Rightarrow$ this energy is "lost" e.g via heating due to resistance, or it may be stored in a capacitor or inductor (solenoid).

- Batteries use chemical process to set up a potential difference to drive currents around circuits.

We can also use <u>induction</u>.

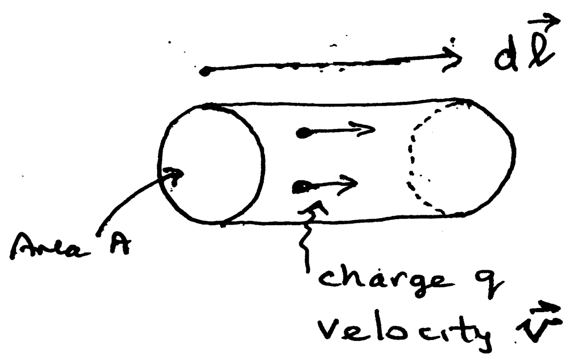
Generators can also be used to drive currents around circuits (in which case mechanical energy is converted to work done moving the charges around the circuit). Generators establish an EMF by a process called **induction**. In fact, most the electricity available via the electric grid is generated by the process of induction.

7.2 Induced currents

It was established experimentally in the first half of the 1800's that there is an intimate link between electricity and magnetism. The first major discovery, In 1820 by the Danish scientist Hans Christian Oersted, occurred when he demonstrated that current-carrying wires produce magnetic fields. This was expressed mathematically by Maxwell in the equation $\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{\vec{\jmath}(\vec{r})}{\epsilon_0 c^2}$. The magnetic field "curls" around the current.

A second major discovery, made in the same year by the French scientist André Marie Ampère, was that wires carrying a current experience a force in a magnetic field. This is a manifestation of the Lorentz force law $\vec{F} = q\,\vec{v} \times \vec{B}$, and is not described by one of Maxwell's equations.

To see how this works, suppose we have a wire with cross-sectional area A carrying a current I. Suppose the current is made up of particles of charge q travelling with velocity $\vec{v}$ parallel to the wire, and that there are N of these charged particles per unit volume. We can determine the force on an element $d\vec{\ell}$ of the wire using the Lorentz force law:



$$d\vec{F} = (NA\,d\ell)\,q\,\vec{v} \times \vec{B},$$

where $NA\,d\ell$ is the number of charges in the element of wire. But since the velocity $\vec{v}$ of the charges is parallel to $d\vec{\ell}$, we can write this as

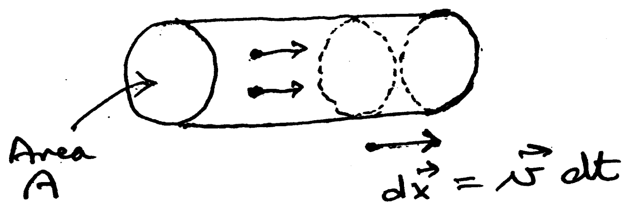
$$d\vec{F} = (NA\,q\,v)\,d\vec{\ell} \times \vec{B}.$$

We will show this is equivalent to

$$d\vec{F} = I\,d\vec{\ell} \times \vec{B}.$$

The current is the amount of charge passing a given point per unit time, which clearly depends on v. In time dt, all of the charge within a distance $dx = v\,dt$ will pass through the end of the element of wire, and that charge is $NA\,dx = NA\,v\,dt$:

q q



So the amount of charged per unit time, or the current, passing through the end of the element of wire is $I = \frac{NAv\,dt}{dt} = NAv$. This completes the proof that the force on the element of wire is

$$d\vec{F} = I\, d\vec{\ell} \times \vec{B}.$$

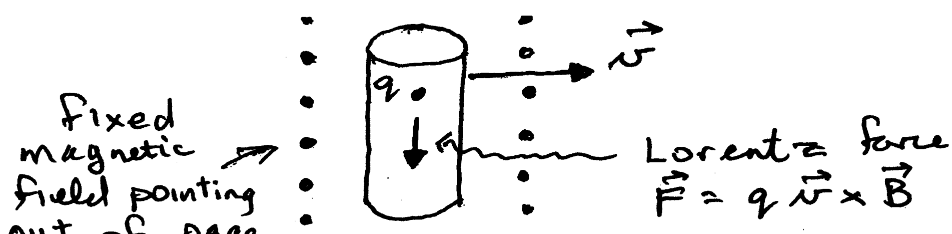
To determine the force on a finite wire, we would integrate this along the whole wire:

$$\vec{F} = I \int d\vec{\ell} \times \vec{B}(\vec{r}).$$

Note that this depends only on the total current, and not on the charges q of the individual particles making up the current.

With the discovery that electric currents produce magnetic fields and are subject to forces in magnetic fields, people began to wonder if magnetic fields could produce electric currents. It was the English scientist Michael Faraday who proved experimentally that this was indeed the case, but only when something was *changing* (or time dependent). Faraday showed that if a wire is moving in a stationary magnetic field, a current is induced in the wire; and if a stationary wire is placed in a changing magnetic field, then a current is also induced in the wire - the changing magnetic field can be produced by moving a magnet, or by varying the current in a wire (therefore varying the magnetic field produced by the wire.

The fact that a wire moving in a stationary magnetic field can induce a current can easily be understood using the Lorentz force law, as illustrated in the example below:



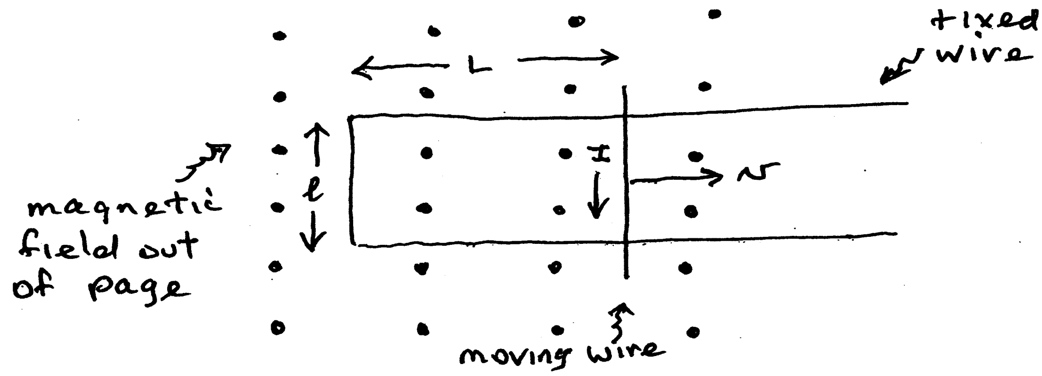
However, the fact that a changing magnetic field can induce a current in a stationary wire is a new phenomenon, and required the introduction of the Maxwell equation

$$\vec{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t},$$

aptly known as Faraday's law. In the case of electrostatics and magnetostatics (no moving charges or changing magnetic fields), it becomes $\vec{\nabla} \times \vec{E}(\vec{r}, t) = 0$.

7.3 The induced EMF due to a wire moving through a static magnetic field

Even though we are considering a static magnetic field, we have moving charges (the charges in the wire are moving as the wire moves). Consider the situation shown in the diagram below: The force on a charge q in the moving portion of the wire is the



Lorentz force $\vec{F} = q\vec{v} \times \vec{B}$ (there is no force due to the magnetic field on the charges in the stationary parts of the wire). In the case of the geometry shown, this force is of magnitude qvB and points downwards, and so will set up a current I in the direction shown. Recalling that the EMF is the work per unit charge done by the force, and that the force actions only in the moving portion of the wire,

$$\mathcal{E} = \frac{1}{q}\int \vec{F}\cdot d\vec{\ell} = vB\ell.$$

But the speed of the wire is $v = \frac{dL}{dt}$, so

$$\begin{aligned}\mathcal{E} &= \frac{dL}{dt}B\ell \\ &= \frac{d}{dt}(BL\ell) \\ &= \frac{d}{dt}(B\,A),\end{aligned}$$

where A is the area enclosed by the current loop. But since $\vec{B}$ is perpendicular to the current loop, $B\,A$ is just the flux $\int \vec{B}(\vec{r})\cdot d\vec{S}$ of the magnetic field through the loop. So:

$$\text{induced EMF } \mathcal{E} = \frac{d}{dt}(\text{magnetic flux through the current loop}).$$

Though proven only for a particular geometry above, this is a general result.

Note that the direction of the induced current is such that the magnetic field it produces gives a change in the magnetic flux through the loop that *opposes* the change in the magnetic flux through the loop due to the motion of the wire - it introduces a flux in the opposite direction. **This is known as Lenz's law**, and allows us to easily deduce the direction of the induced EMF. *The direction of the induced EMF by changing magnetic flus through a loop is always such that the magnetic field produced by the induced current opposes the change in magnetic flux - nature "resists" the changing flux.*

7.4 Stationary circuit, changing magnetic field

In this case, the induced EMF cannot be explained or computed using an argument based on the Lorentz force law. This is a new physical principle. It was Faraday who discovered experimentally that if the magnetic field in a region of space is changing with time, then electric fields are induced in that region of space. If a wire is present, it is this electric field which provides a force on the charges to induce a current. However, even in the absence of a wire, the electric fields is produced - this is very important for the phenomenon of electromagnetic waves. Maxwell showed that Faraday's results could be summarised in the vector equation

$$\vec{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t},$$

which is known and Faraday's law. The electric field induced by a time varying magnetic field is given on the left-hand side of the equation (recall that in magnetostatics, the right hand side is zero as we only consider static magnetic fields, in which case $\vec{\nabla} \times \vec{E}(\vec{r}) = 0$).

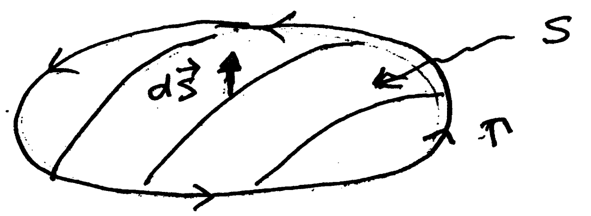
If we have a closed loop of wire, let Γ denote the closed path it determines. The EMF induced by the electric field produced by the changing magnetic field is

$$\mathcal{E} = \oint_\Gamma \vec{E} \cdot d\vec{\ell}.$$

If S is any two dimensional surface whose boundary is Γ, Gauss's law tells us

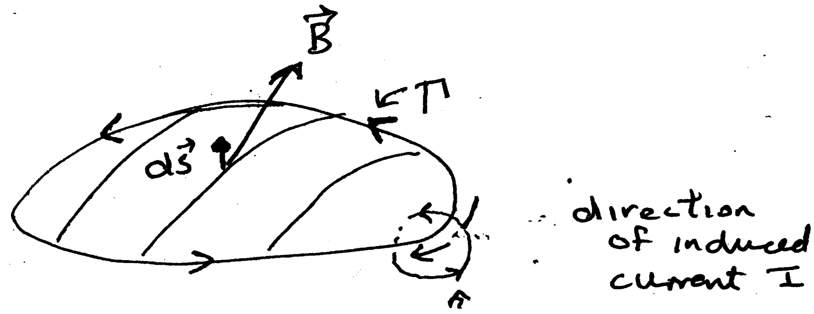
$$\mathcal{E} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S}.$$

Using Faraday's law, this becomes



$$\mathcal{E} = -\int_S \frac{d\vec{B}}{dt} \cdot d\vec{S} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S} = -\frac{d}{dt}(\text{magnetic flux through } S).$$

Note that the minus sign in this expression relates to Lenz's law: if $d\vec{S}$ are chosen to point out of the page; then the loop Γ is *counter-clockwise* (by the right hand rule). If the magnetic field also points out of the page and is increasing, then the induced current is *clockwise.* Again, the direction of the induced current is such that the magnetic field it produces opposes the change in the flux through the loop - it gives a flux in the opposite direction.



So, in both the case of a wire moving through a static magnetic field, and a static wire in the presence of a changing magnetic field, the same result applies for the induced EMF:

$$|\mathcal{E}| = |\frac{d}{dt}(\text{magnetic flux through circuit})|,$$

with the direction of the induced EMF determined by Lenz's law. In the former case, we can explain the result using the Lorentz force law with no need to use Maxwell's equations; in the latter case, we have to introduce a new term into one the Maxwell equations that we used for electrostatics and magnetostatics, namely

$$\vec{\nabla} \times \vec{E}(\vec{r}) = 0 \quad \Rightarrow \quad \vec{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t}.$$

There is no apparent explanation as to why the induced EMF is given by the same formula in both cases. This "coincidence" was one of the puzzles that ultimately led Einstein to the Special Theory of Relativity (where we can find an explanation).

7.5 Completing Ampere's law - the "displacement current"

Up until now, we have been using a version of Ampere's law applicable to magnetostatics (time-independent charge and current densities which means time-independent electric and magnetic fields):

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{1}{\epsilon_0 c^2} \vec{j}(\vec{r}).$$

However, once we introduce time dependence, an additional term called the "displacement current" must be included in Ampere's law.

To understand why we need to modify Ampere's law, let's go back an reconsider an analysis we did earlier. We considered a closed surface S enclosing a volume V. The net flow of current through the closed surface S is

$$\begin{aligned}\oint_S \vec{j}(\vec{r}) \cdot d\vec{S} &= \epsilon_0 c^2 \oint_S (\vec{\nabla} \times \vec{B}(\vec{r})) \cdot d\vec{S} \quad \text{(using Ampere's law)} \\ &= \int_V d^3\vec{r}\, \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}(\vec{r})) \quad \text{(using Gauss's law)} \\ &= 0\end{aligned}$$

as a result of the vector identity $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}(\vec{r}))$ for any vector field $\vec{V}(\vec{r})$. The fact that the net flow of current through any closed surface S is zero is consistent with the