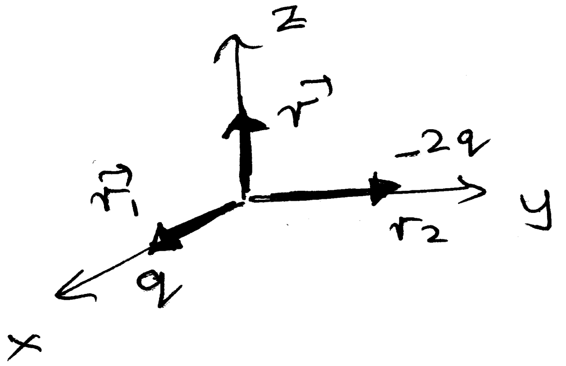
EM PROBLEM SET 4 2019

SOLUTIONS

1.



The key formula is that for a charge q located at $\vec{r}_0$, the electric potential at a point $\vec{r}$ is

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0|\vec{r}-\vec{r}_0|}.$$

In the case here, $\vec{r} = (0, 0, c)$,

and $V(\vec{r}) = \dfrac{q}{4\pi\varepsilon_0|\vec{r}-\vec{r}_1|}$ ← electric potential due to q

$+ \dfrac{(-2q)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_2|}$ ← electric potential due to charge $-2q$

$\vec{r} = (0, 0, c)$

$\vec{r}_1 = (a, 0, 0)$

$\vec{r}-\vec{r}_1 = (-a, 0, c)$

$|\vec{r}-\vec{r}_1| = \sqrt{(-a)^2+0+c^2} = \sqrt{a^2+c^2}$

$$\vec{r} = (0, 0, c)$$

$$\vec{r_2} = (0, b, 0)$$

$$\vec{r} - \vec{r_2} = (0, -b, c)$$

$$|\vec{r} - \vec{r_2}| = \sqrt{0 + (-b)^2 + c^2} = \sqrt{b^2 + c^2}$$

So
$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0\sqrt{a^2+c^2}} - \frac{2q}{4\pi\varepsilon_0\sqrt{b^2+c^2}}$$

2, $\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$

$$\vec{\nabla} V(\vec{r}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(\alpha z - \beta x y)$$

$$= (-\beta y, -\beta x, \alpha)$$

$$\Rightarrow \vec{E}(\vec{r}) = (\beta y, \beta x, -\alpha)$$

$$= \beta y\, \vec{e}_x + \beta x\, \vec{e}_y - \alpha\, \vec{e}_z$$

3. $\vec{\nabla}\phi(\vec{r})$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(x^3+y^3+z^3)$$

$$= (3x^2, 3y^2, 3z^2)$$

$$= 3x^2\vec{e}_x + 3y^2\vec{e}_y + 3z^2\vec{e}_z$$

4. $\vec{\nabla}\cdot\vec{A}(\vec{r})$

$$= (\partial_x, \partial_y, \partial_z)\cdot(A_x(\vec{r}), A_y(\vec{r}), A_z(\vec{r}))$$

$$= \partial_x A_x(\vec{r}) + \partial_y A_y(\vec{r}) + \partial_z A_z(\vec{r})$$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(-xy)$$

$$= z + 0 + 0$$

$$= z$$

5. $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$\vec{A}(\vec{r}) = (xz, z, -xy).$

Method 1:

$$(\vec{\nabla}\times\vec{A}(\vec{r}))_x = \frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y$$

$$= \frac{\partial}{\partial y}(-xy) - \frac{\partial}{\partial z}(z)$$

$$= -x - 1$$

$$\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)_y = \frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z$$

$$= \frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(-xy)$$

$$= x - (-y)$$

$$= x + y$$

$$\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)_z = \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x$$

$$= \frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(xz)$$

$$= 0 - 0$$

$$= 0$$

So $$\vec{\nabla}\times\vec{A}(\vec{r}) = -(x+1)\,\vec{e}_x + (x+y)\,\vec{e}_y$$

Method 2:

$$\begin{array}{cc|cccccc|c}
\vec{\nabla}: & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\vec{A}: & x & z & z & -xy & xz & z & -xy
\end{array}$$

$$\left(\vec{\nabla}\times\vec{A}\right)_x = \frac{\partial}{\partial y}(-xy) - \frac{\partial}{\partial z}(z) = -x-1$$

$$\left(\vec{\nabla}\times\vec{A}\right)_y = \frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(-xy) = x-(-y) = x+y$$

$$\left(\vec{\nabla}\times\vec{A}\right)_z = \frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(xz) = 0$$