# Conductors

- In an insulator – all electrons are bound and not free to move.
- In a conductor, some of the electrons (conduction electrons) are free to move.

For a conductor:

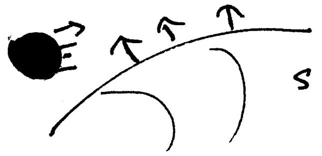
– the $\vec{E}$ field inside the conductor is zero – otherwise conduction electrons would experience a force and move.

If a conductor is placed in an external $\vec{E}$ field: the charges inside the conductor move to create an internal $\vec{E}$ field that cancels the external one

$\Rightarrow$ $\vec{E} = 0$ inside the conductor.

– Any net charge on the conductor resides on the surface. This follows from $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$, $\vec{E} = 0$ inside $\Rightarrow$ charge density $\rho = 0$ inside.

– The electric field just above the surface of a conductor is perpendicular to the surface: any component of $\vec{E}$ parallel to the surface would create a surface current.

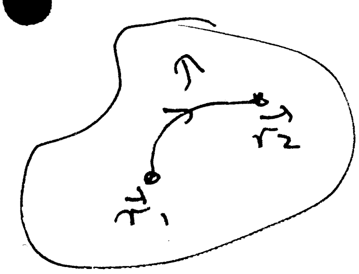



– A conductor is an equipotential.

Reason: if $\vec{r}_1$ and $\vec{r}_2$ are two points inside the conductor

$$V(\vec{r}_2) - V(\vec{r}_1) = \int_{\mathcal{P}} \vec{E}(\vec{r})\cdot d\vec{l}$$

path joining $\vec{r}_1$ and $\vec{r}_2$ inside conductor




$= 0$ since $\vec{E} = 0$.

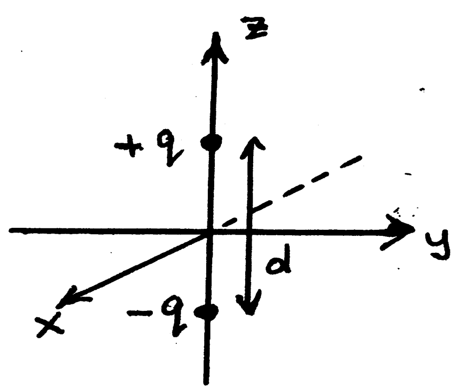
# 5 Dipole and multipole electric fields

## 5.1 Dipole electric field

Consider a pair of charges $+q$ and $-q$ separated by a distance $d$ (where $q$ is assumed positive).This is called an electric **dipole**. Although the total charge is zero, the electric field is nonzero due to the separation of the charges.

We cannot use Gauss's law in this case to determine the resulting electric field. Instead we will determine the electric potential $V$, which is the sum of the electric potentials produced by the individual charges; the electric field is then computed via the gradient of the electric potential, $\vec{E} = -\vec{\nabla} V$.

We will choose a cartesian coordinate system whose origin is midway between the two charges, and in which the charges lie on the $z$-axis: the charge $+q$ at the point $(0, 0, +\frac{d}{2})$ (in coordinates $(x, y, z)$), and the charge $-q$ at the point $(0, 0, -\frac{d}{2})$.




Then, using the result that the electric potential at a point $\vec{r}$ due to a point charge $q$ at point $\vec{r}_0$ is

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_0|},$$

the electric potential at a point $\vec{r}$ due to the dipole is

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(x^2 + y^2 + (z - d/2)^2}} - \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + (z + d/2)^2}}. \tag{5.1}$$

Suppose we are computing the electric potential at a point $\vec{r}$ whose distance $r = \sqrt{x^2 + y^2 + z^2}$

from the origin is large compared to the distance between the charges, $\frac{d}{r} << 1$. Then

$$\begin{aligned}
x^2 + y^2 + (z \mp d/2)^2 &= x^2 + y^2 + z^2 \mp z\,d + \frac{d^2}{4} \\
&= r^2 \pm z\,d + \frac{d^2}{4} \\
&= r^2\Big(1 \mp \frac{z}{r}\frac{d}{r} + \frac{d^2}{4r^2}\Big) \\
&\approx r^2\Big(1 \mp \frac{z\,d}{r^2}\Big), \qquad (5.2)
\end{aligned}$$

since $\frac{d^2}{4r^2} << \frac{d}{r}$ if $\frac{d}{r} << 1$. Next, using $\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} \approx 1 - \frac{1}{2}x$ when $|x| << 1$,

$$\frac{1}{\sqrt{x^2 + y^2 + (z \mp d/2)^2}} \approx \frac{1}{r}\Big(1 \pm \frac{z\,d}{2r^2}\Big). \qquad (5.3)$$

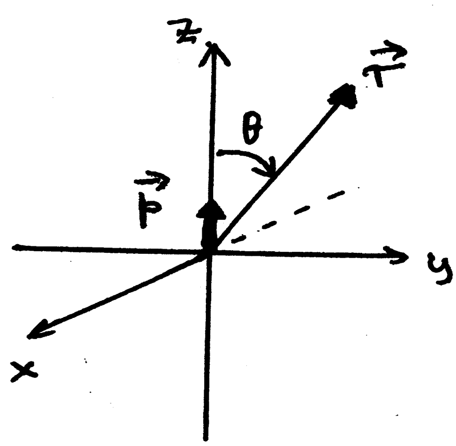
Finally, using $\frac{z}{r} = \cos\theta$, where $\theta$ is the angle between the position vector $\vec{r}$ and the $z$-axis,

$$\frac{1}{\sqrt{x^2 + y^2 + (z \mp d/2)^2}} \approx \frac{1}{r}\Big(1 \pm \frac{d\cos\theta}{2r}\Big).$$

Substituting back into (5.8),

$$V(\vec{r}) = \frac{q\,d}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}.$$

Define a vector $\vec{p}$ of magnitude $q\,d$, called the **electric dipole moment**, which points in the direction from the charge $-q$ to $+q$ (so in our choice of coordinates, in the $+z$ direction).




Then $q\,d\,r\cos\theta = \vec{p}\cdot\vec{r}$, and we can write

$$V(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0\,r^3} = \frac{\vec{p}\cdot\vec{e}_r}{4\pi\epsilon_0\,r^2},$$

where $\vec{a}_r$ is the unit vector in the radial direction. Note that unlike a single point charge, for which the electric potential dies off as $\frac{1}{r}$, for the dipole (a pair of separated charges with total charge zero), the electric potential dies off as $\frac{1}{r^2}$ (faster). If the centre of the dipole is at position $\vec{r}_0$, then the result is

$$V(\vec{r}) = \frac{\vec{p}\cdot(\vec{r}-\vec{r}_0)}{4\pi\epsilon_0\,|\vec{r}-\vec{r}_0)|^3}.$$

In principle, to find the shape of the resulting electric field, we could draw equipotentials (two dimensional surfaces on which the electric potential is constant) - the electric field lines are then perpendicular to the equipotential surfaces. In practice, it is easier to just compute the electric field. We will choose the polarization vector to point in the $+z$ direction, $\vec{p} = p\,\vec{e}_z$, in which case $\vec{p}\cdot\vec{r} = p\,z$, and so

$$V(\vec{r}) = \frac{p}{4\pi\epsilon_0}\frac{z}{r^3}.$$

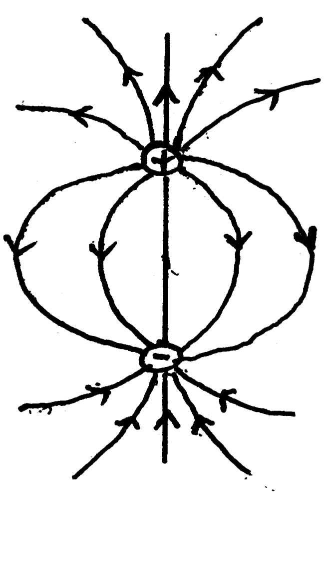
The electric field is $\vec{E} = -\vec{\nabla}V = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z}\right)$. For the $x$ and $y$ components , using $\frac{\partial r}{\partial x} = \frac{x}{r}$ and similarly for $y$, we get

$$E_x = \frac{p}{4\pi\epsilon_0}\frac{3zx}{r^5}, \quad E_y = \frac{p}{4\pi\epsilon_0}\frac{3zy}{r^5}.$$

The $z$ component is more complicated,

$$E_z = -\frac{p}{4\pi\epsilon_0}\frac{1}{r^3} + \frac{p}{4\pi\epsilon_0}\frac{3z^2}{r^5}.$$

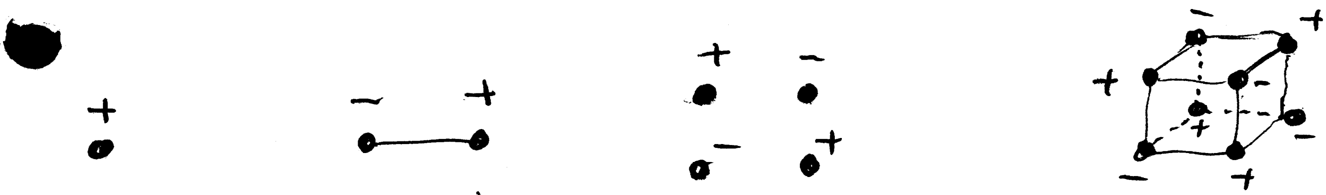
If we plot the field up, we get the classic **dipole electric field**.



5.2

# Monopole, Dipole, Quadrupole and Octupole Charge distributions

The behavior of the electric potential at large distances for "pure" monopole, dipole, quadrupole and octupole charge distributions are:



| Monopole | Dipole | Quadrupole | Octupole |
|---|---|---|---|
| $\phi \sim \frac{1}{r}$ | $\phi \sim \frac{1}{r^2}$ | $\phi \sim \frac{1}{r^3}$ | $\phi \sim \frac{1}{r^4}$ |

Note that except for the monopole (point charge), all of the other charge distribution have total charge zero.

# MAGNETOSTATICS

- Maxwell's equations in full generality:

1. $\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\varepsilon_0}$ (Gauss's law)

   $\rho(\vec{r}, t) =$ charge density

2. $\vec{\nabla} \times \vec{E}(\vec{r}, t) = \frac{\partial \vec{B}(\vec{r}, t)}{\partial t}$ (Faraday's law)

3. $\vec{\nabla} \cdot \vec{B}(\vec{r}, t) = 0$

4. $\vec{\nabla} \times \vec{B}(\vec{r}, t) = \frac{1}{c^2} \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} + \frac{\vec{j}(\vec{r}, t)}{\varepsilon_0 c^2}$

   (Ampere's law)

   $\vec{j}(\vec{r}, t) =$ current density.

Supplemented by the Lorentz force law

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}.$$

- Electrostatics: no current density ($\vec{j} = 0$)

static charge density $\rho(\vec{r}) \Rightarrow \vec{B} = 0$, $\vec{E}$ is static (time indep), $\vec{E}(\vec{r})$

Magnetostatics: steady currents (which involve flow of charge $\Rightarrow$ time independent current density $\Rightarrow$ time independent magnetic fields

So far magnetostatics: + electrostatics

$$\vec{\nabla}.\vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0} \qquad \text{(Gauss's law)}$$

$$\vec{\nabla}\times\vec{E}(\vec{r}) = 0$$

$$\vec{\nabla}.\vec{B}(\vec{r}) = 0$$

$$\vec{\nabla}\times\vec{B}(\vec{r}) = \frac{\vec{J}(\vec{r})}{\varepsilon_0 c^2} \qquad \text{(Ampere's law).}$$

- <u>Significance of $\vec{\nabla}.\vec{B} = 0$</u>

• Applies for any $\vec{B}$ field, no matter how it is produced.

It means the flux of $\vec{B}$ through any closed surface S is always zero:

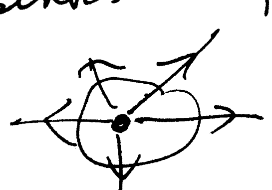
- 


Gauss's theorem

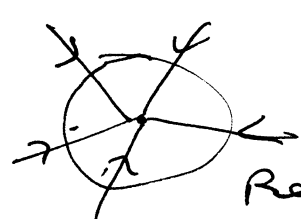
$$\oint_S \vec{B}.d\vec{s} = \int_V \underbrace{\vec{\nabla}.\vec{B}}_{=0}\, d^3\vec{r}$$

$$= 0$$

<u>So</u>: there can be no <u>isolated</u> "source" or "sinks" of the $\vec{B}$ field



Requires $\vec{\nabla}.\vec{B} > 0$ (diverging field)

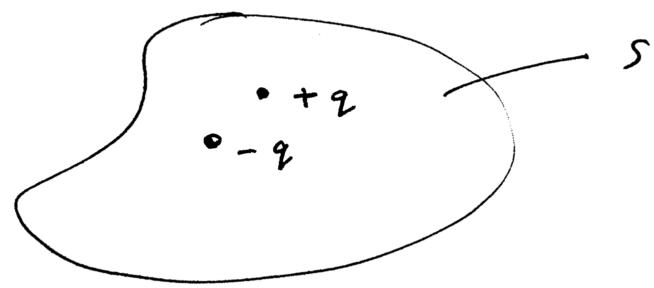


Requires $\vec{\nabla}.\vec{B} < 0$ (converging field)

So : isolated magnetic charges cannot exist (or magnetic monopoles cannot exist).

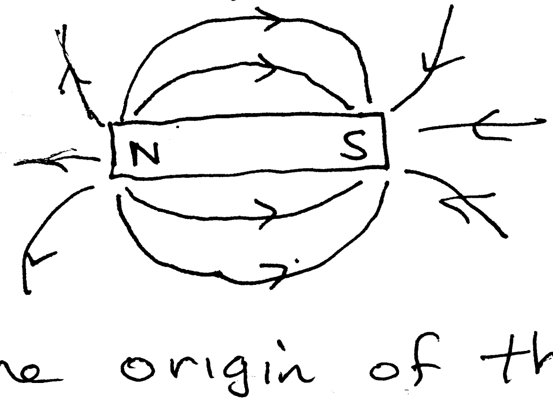
[For $\vec{E}$, $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$, can have sources + sinks for the $\vec{E}$ field, isolated electric charges can exist].

• But: we saw that for an electric dipole, $\vec{\nabla}.\vec{E} = 0$ (no charge inside a surface enclosing the dipole).



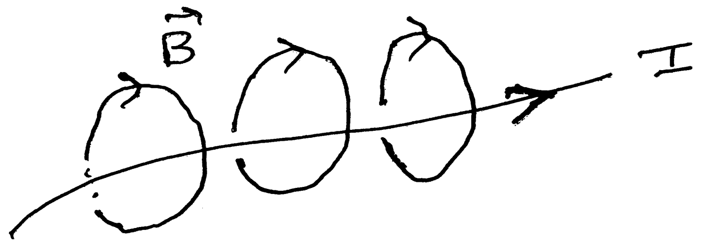

• For magnetic fields, a dipole field is the simplest field that can exist (it satisfies $\vec{\nabla}.\vec{B} = 0$)

e.g a bar magnet



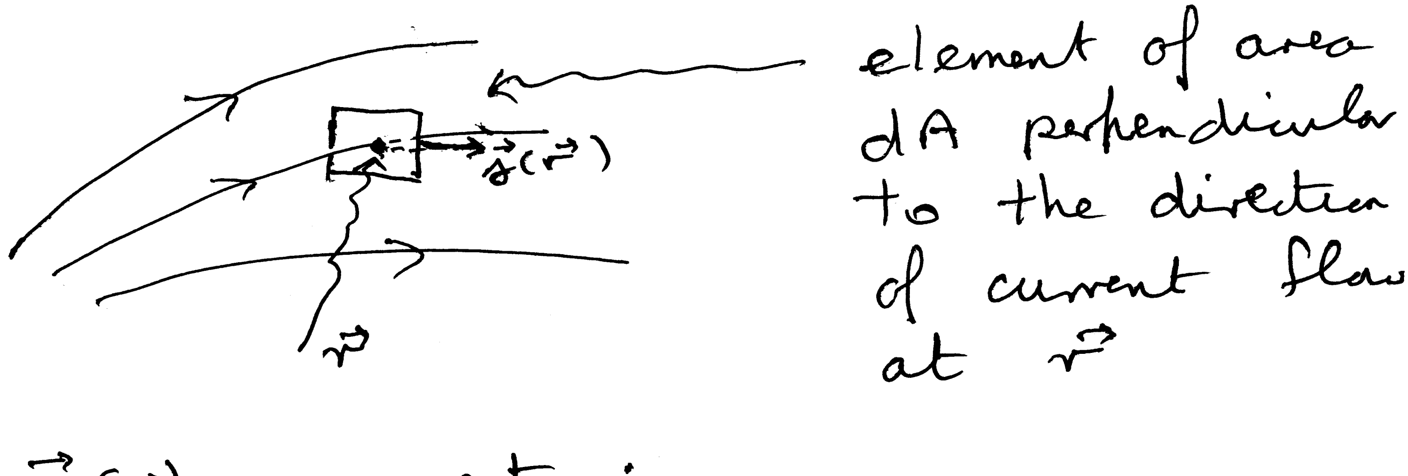

However: the origin of this dipole field is not from isolated magnetic charges – it is due to the microsopic current loops created by electrons as they orbit nuclei.

- Macroscopic (large scale) currents are also another commen source of magnetic fields




## Current density

Consider a region in which current is flowing:

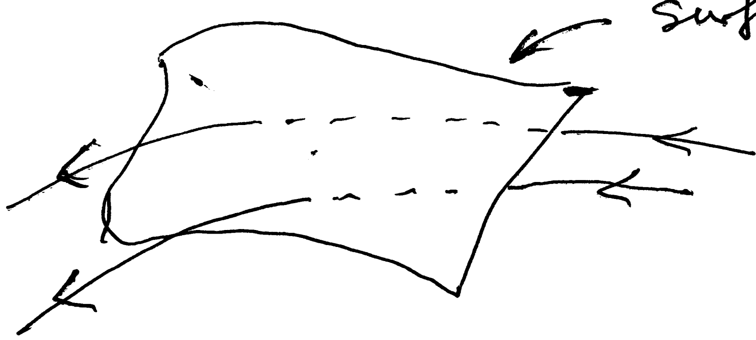



$\vec{j}(\vec{r})$ is a vector:
- the direction of $\vec{j}(\vec{r})$ is the direction of the current flow through $\vec{r}$

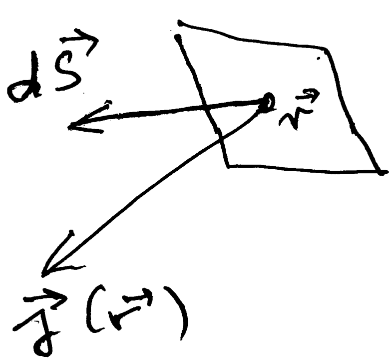
$$-\,|\vec{j}(\vec{r})| = \frac{\text{current through area } dA}{dA}$$

= current per unit cross-sectional area at $\vec{r}$ (units are amps/$m^2$).

- Consider a surface S with a current flowing through it




For an element $d\vec{S}$ of surface




$d\vec{S}$: vector perpendicular to surface element, $|d\vec{S}|$ = area

$\vec{j}(\vec{r})$: current density.

Current through surface element

$$= \underbrace{\vec{j}(\vec{r}) \cdot d\vec{S}}$$

the dot product projects the piece of $\vec{j}$ perpendicular to surface

The piece of $\vec{j}$ parallel to the surface does not contribute to the current through $dS$ (and does not contribute to $\vec{j}(\vec{r})\cdot d\vec{S}$).

• So the current through a finite surface $S$ is

$$\underbrace{\int_S \vec{j}(\vec{r})\cdot d\vec{S}}_{\substack{\text{current through } S \\ \equiv \text{ flux of } \vec{j} \text{ through } S.}}$$

• <u>Ampere's law</u>:

$$\vec{\nabla}\times\vec{B}(\vec{r}) = \frac{\vec{j}(\vec{r})}{\varepsilon_0 c^2}$$

Given a closed curve $\Gamma$


the <u>circulation of a magnetic field $\vec{B}(\vec{r})$ around</u> $\Gamma$ is

$$\oint_{\Gamma} \vec{B}(\vec{r}) \cdot d\vec{l} ,$$

as $\vec{B}(\vec{r}) \cdot d\vec{l}$ projects out the component of the magnetic field at point $\vec{r}$ on $\Gamma$ in the direction $d\vec{l}$ of the tangent to the curve. The component of $\vec{B}$ perpendicular to $\Gamma$ does not contribute to $\vec{B}(\vec{r}) \cdot d\vec{l}$

- Let $S$ be any surface with boundary the closed curve $\Gamma$;


Then
$$\oint_{\Gamma} \vec{B}(\vec{r}) \cdot d\vec{l}$$
$$= \int_S (\vec{\nabla} \times \vec{B}(\vec{r})) \cdot d\vec{S} \quad \text{by Gauss's theorem}$$
$$= \frac{1}{\varepsilon_0 c^2} \underbrace{\int_S \vec{j}(\vec{r}) \cdot d\vec{S}}_{\text{current through } S}$$

• So : Ampere's law tells us

$\boxed{\text{circulation of } \vec{B} \text{ around a closed curve } \Gamma = \frac{1}{\varepsilon_0 c^2} (\text{current through a surface } S \text{ with boundary } \Gamma)}$

• Note : $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow$ a $\vec{B}$ field does not divergerge or converge.

But $\vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\varepsilon_0 c^2}$

$\Rightarrow$ a magnetic field "curls" around a current (or has a net circulation $\int_\Gamma \vec{B}(\vec{r}) \cdot d\vec{l}$ around a current).