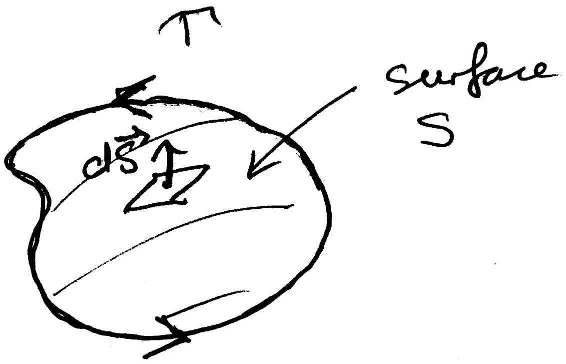
Ampere's law

$$\underbrace{\vec{\nabla} \times B(\vec{r})}_{\text{curl of } \vec{B}} = \frac{1}{\varepsilon_0 c^2} \vec{J}(\vec{r})$$

current density

Recall: $\vec{\nabla} \times B(\vec{r})$ tells us about the circulation of $\vec{B}$ around a closed curve via Stokes' theorem:

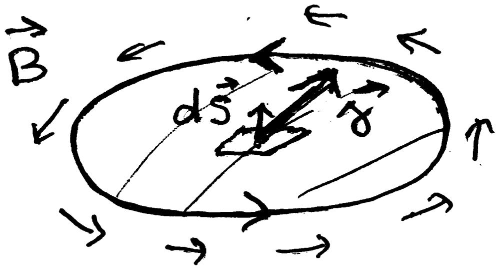


$$\underbrace{\oint_\Gamma \vec{B}(\vec{r}) \cdot d\vec{l}}_{\substack{\text{circulation} \\ \text{of } \vec{B} \text{ around} \\ \Gamma}} = \int_S (\vec{\nabla} \times \vec{B}(\vec{r})) \cdot d\vec{S}$$

Stokes' theorem

$$= \frac{1}{\varepsilon_0 c^2} \underbrace{\int_S \vec{J}(\vec{r}) \cdot d\vec{S}}_{\substack{\text{current} \\ \text{through surface} \\ S.}}$$

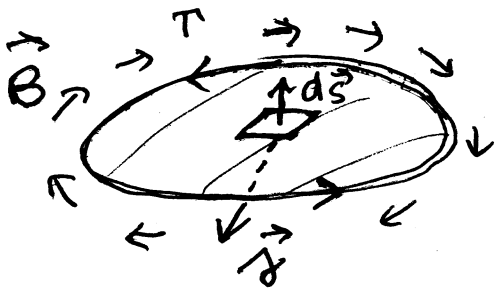
Ampere's law



$\vec{j}(\vec{r}) . ds > 0$

$\Rightarrow$ "positive" circulation around T

$\Rightarrow$ $\vec{B}$ is in direction of T (given by right hand rule)



$\vec{j}(\vec{r}) . d\vec{s} < 0$

$\Rightarrow$ "negative" circulation around T

$\Rightarrow$ $\vec{B}$ is in opposite direction to T (given by right hand rule)

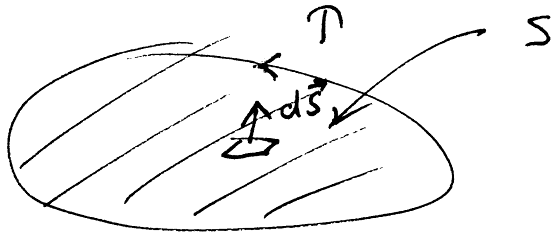
So the magnetic field "circulates" or "curls" around the direction of the current.

(For electrostatics $\vec{\nabla} \times \vec{E} = 0$, no circulation possible. But $\vec{\nabla} . \vec{E} = \frac{\rho}{\varepsilon_0}$, field can <u>diverge</u> or converge. For $\vec{B}$ field, $\vec{\nabla} . \vec{B} = 0$, no convergence or divergence).

Applying Ampere's law

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{\vec{J}(r)}{\varepsilon_0 c^2}$$

Consider a surface S with boundary Γ:



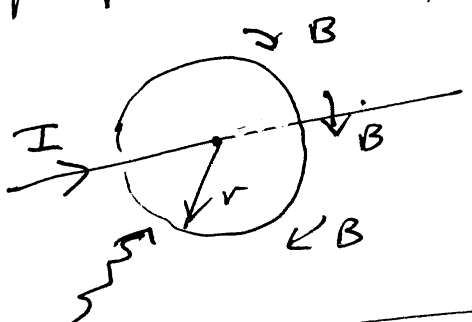
- $$\underbrace{\frac{1}{\varepsilon_0 c^2} \int_S \vec{J}(\vec{r}) \cdot d\vec{S}}_{\text{current through } S} = \int_S (\vec{\nabla} \times \vec{B}(\vec{r})) \cdot d\vec{S} \quad \leftarrow \text{Ampere's law}$$

$$= \underbrace{\oint_\Gamma \vec{B}(\vec{r}) \cdot d\vec{l}}_{\text{circulation of } \vec{B} \text{ around } \Gamma}$$

Example 1

- . $\vec{B}$ field due to a straight wire

Take S to be the interior of a circle of radius r surrounding the wire, S perpendicular to direction of current.



Amperian loop

Ampere's law ⇒

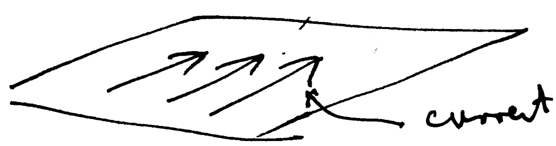
$$\frac{1}{\varepsilon_0 c^2} I = \oint \vec{B}(\vec{r}) \cdot dl$$

$$= 2\pi r \, B(r)$$

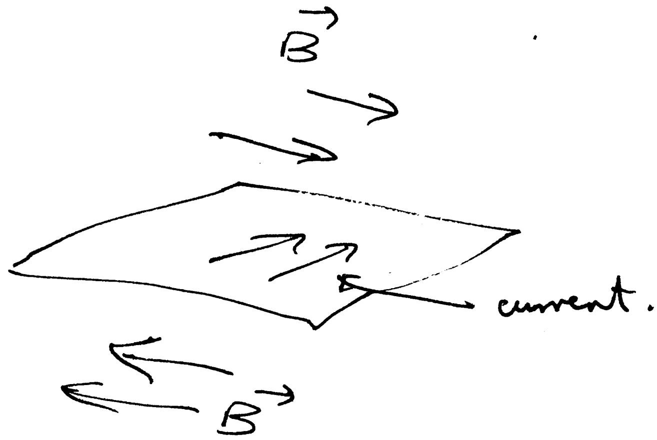
$$\Rightarrow \boxed{B(r) = \frac{1}{2\pi\varepsilon_0 c^2} \frac{I}{r}}$$

Example 2:

$\vec{B}$ field due to an infinite plane of current

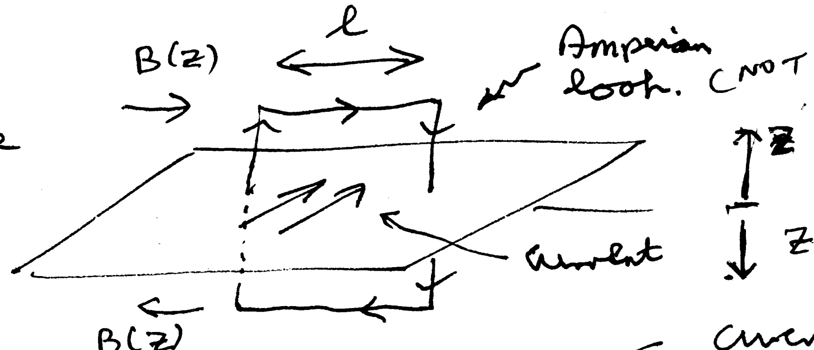


Symmetry $\Rightarrow$



Ampere's law $\oint_\Gamma \vec{B} \cdot d\vec{l} =$ current enclosed in surface S with boundary Γ

Take



current per unit length $\perp$ to current.

Current enclosed $= \lambda \ell$

$$\oint_\Gamma \vec{B} \cdot d\vec{l} = 2 B(z) . \ell$$

one $B(z) . \ell$ from above plane, one from below

$$2 B(z) \ell = \frac{\lambda \ell}{\epsilon_0 c^2}$$

$$\Rightarrow B(z) = \frac{\lambda}{2 \epsilon_0 c^2}$$

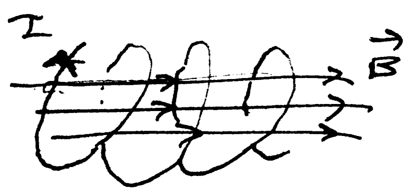
, independent of z.

Solenoids (or inductors)

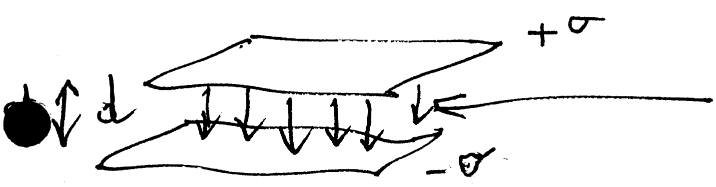
← a current carrying wire tightly coiled on the surface of an imaginary cylinder.

Result: a constant $\vec{B}$ field

- along axis of cylinder.



It is the magnetic analogue of a parallel plate capacitor

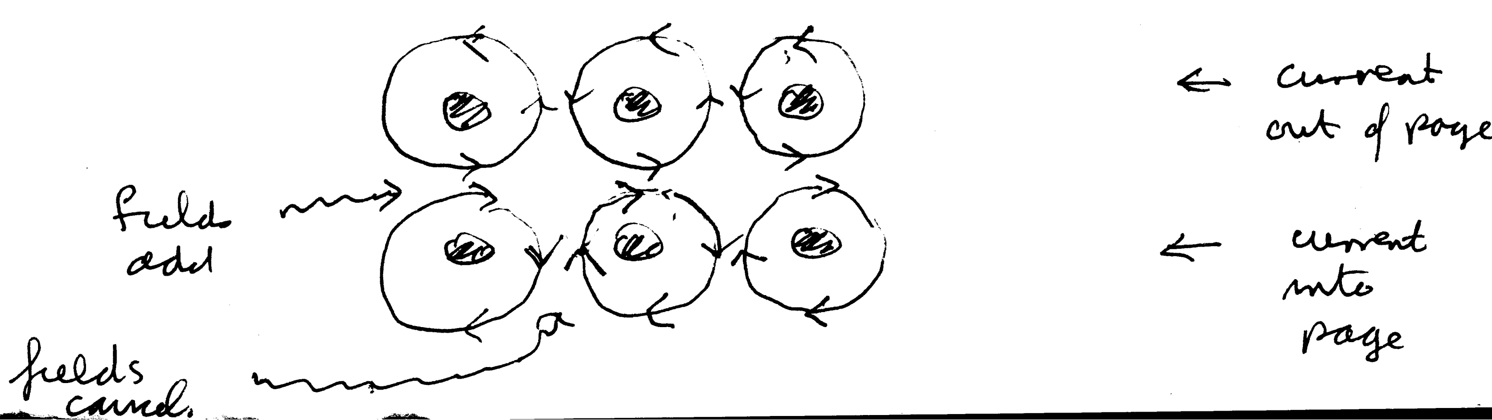


$\vec{E}$ constant

$|\vec{E}| = \frac{\sigma}{\varepsilon_0}$ d

- $\vec{B}$ Field of solenoid:

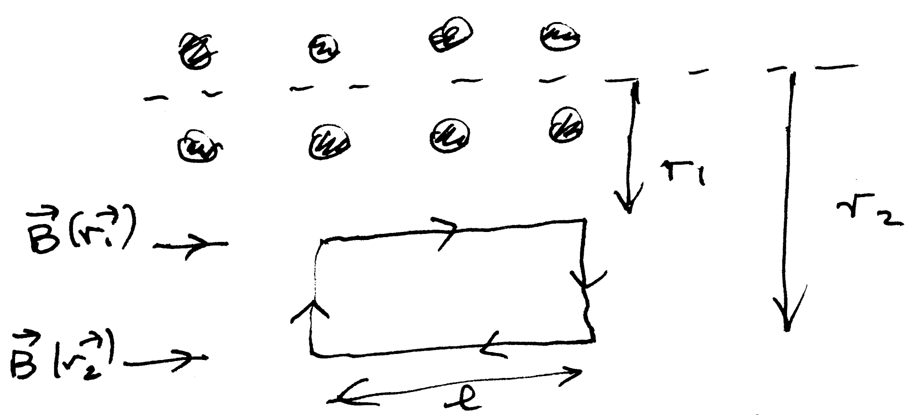
Side view of solenoid



- Radial components of fields cancel between adjacent wires ⇒ only field parallel to axis can exist.

<u>Also</u>: since the $\vec{B}$ field of a wire decreases with distance, the field of the solenoid → 0 at infinite radial distance from axis of solenoid.

- Consider the following amperian loop



Current enclosed by loop = 0

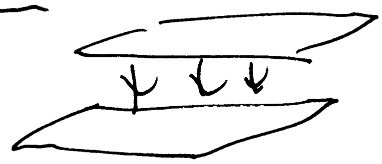
$$\Rightarrow \oint \vec{B}\cdot d\vec{l} = 0$$

No radial field ⇒ ~~B~~ only horizontal sides of loop contribute.

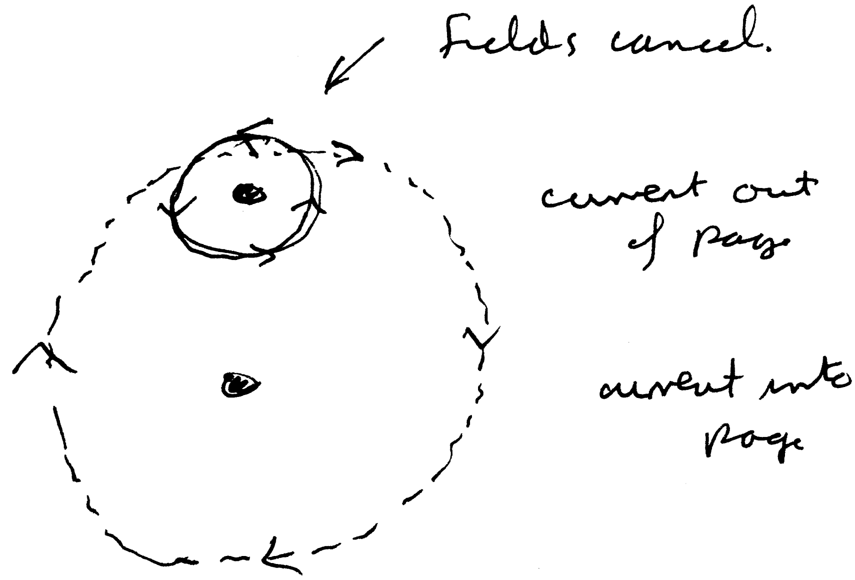
$$\Rightarrow B(r_1)\, l - B(r_2)\, l = 0$$

$$\Rightarrow B(r_1) = B(r_2).$$

But as $r_2 \to \infty$, $B(r_2) = 0$ ⇒ $\boxed{B(r_1) = 0}$

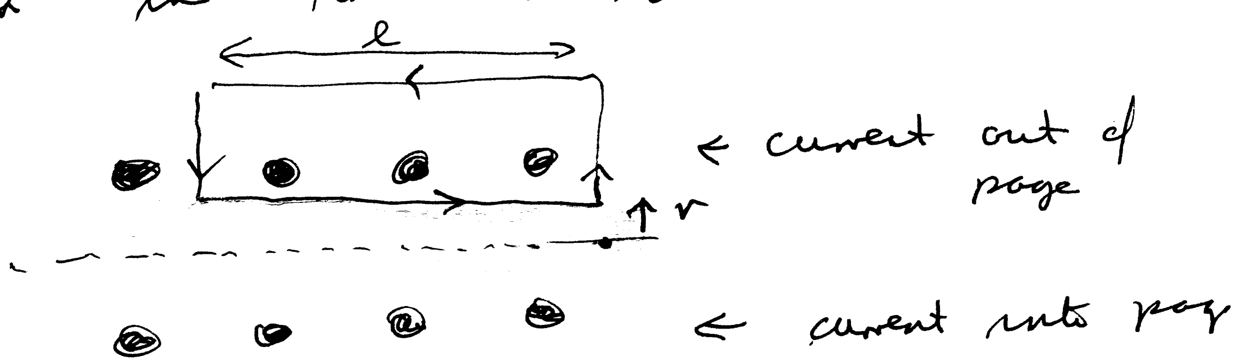
Remarkable result - <u>no $\vec{B}$ field outside</u>
the <u>infinite solenoid</u>
(like a capacitor  , no

$\vec{E}$ field out side the region between plates)



Reason :

• $\vec{B}$ field inside solenoid.
Suppose there are n turns per unit length in the solenoid



$$\oint \vec{B} \cdot d\vec{l} = \frac{\text{current enclosed}}{\varepsilon_0 c^2} = \frac{n l I}{\varepsilon_0 c^2}$$

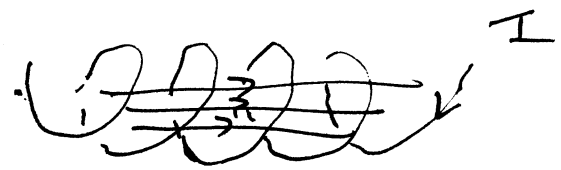
$\vec{B}$ vanishes except on the part inside solenoid

$$\Rightarrow \oint \vec{B}.d\vec{l} = B(r).l$$

$$\Rightarrow B(r)\, l = \frac{n l I}{\varepsilon_0 c^2}$$

$$\boxed{B(r) = \frac{nI}{\varepsilon_0 c^2}}$$ (independent of r)

Co constant field inside solenoid.



Magnetic forces do no work on charged particles

- Lorentz force law

$$\vec{F}(\vec{r}) = q\,\vec{E}(\vec{r}) + q\,\vec{v}(\vec{r}) \times \vec{B}(\vec{r})$$

↑ force on a particle of charge q in an $\vec{E}$ and $\vec{B}$ field.

- For a particle in an $\vec{E}$ field:

$$F(\vec{r}) = q\vec{E}(\vec{r})$$

⇒ work to move particle from $\vec{r}$ to $\vec{r}+d\vec{r}$

is $\quad W = -q\,\vec{E}(\vec{r}) \cdot d\vec{r}$

This allowed us to define the potential energy of the particle in the $\vec{E}$ field, and then define the electric potential

$$V(\vec{r}) = \frac{\text{potential energy}}{q}.$$

and $\quad V(\vec{r}_2) - V(\vec{r}_1) = \dfrac{\text{work to move from } \vec{r}_1 \text{ to } \vec{r}_2}{q}.$

This is summarised in the relationship

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r}).$$

- Consider a charged particle moving in a magnetic field

$$\vec{F}(\vec{r}) = q\,\vec{v}(\vec{r}) \times \vec{B}(\vec{r}).$$

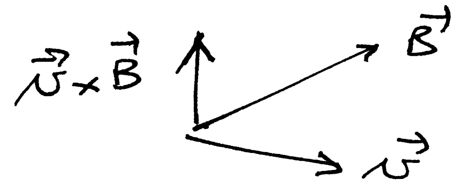
In time t, the particle will undergo a displacement $d\vec{r} = \vec{v}(\vec{r})\, dt$

$\Rightarrow$ work done by the magnetic force

$$= F(\vec{r}) \cdot d\vec{r}$$

$$= q\left(\vec{v}(\vec{r}) \times \vec{B}(\vec{r})\right) \cdot \vec{v}(\vec{r})\, dt$$

$$= 0 \quad \text{since } \vec{v} \times \vec{B} \perp \vec{v}$$



- e.g. an electric motor or a generator.

Require a magnetic field, but NO ENERGY IS EXTRACTED FROM THE MAGNETIC FIELD.

- Energy is provided by electric currents for a motor, or by turning the shaft of a generator.

- the magnetic field <u>mediates</u> conversion of the input to the output (electricity $\rightarrow$ work in a motor, work $\rightarrow$ electricity in a generator).

The Vector Potential

- One of Maxwell's equations is

$$\vec{\nabla}.\vec{B}(\vec{r}) = 0$$

($\Rightarrow$ no isolated magnetic charges via Gauss's theorem).

We can solve this by writing

- $$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$$

← vector potential

as $\vec{\nabla}.(\vec{\nabla}\times\vec{A}(\vec{r})) = 0$ always (see practice problems

[$\vec{\nabla}\times\vec{A}(\vec{r}) = (\partial_y A_z - \partial_z A_y, \partial_z A_x - \partial_x A_z, \partial_x A_y - \partial_y A_x)$

- $$\begin{aligned}\vec{\nabla}.(\vec{\nabla}\times\vec{A}(\vec{r})) = &\ \partial_x(\cancel{\partial_y A_z} - \cancel{\partial_z A_y}) \\ &+ \partial_y(\cancel{\partial_z A_x} - \cancel{\partial_x A_z}) \\ &+ \partial_z(\cancel{\partial_x A_y} - \cancel{\partial_y A_x}) \\ = &\ 0 \quad]\end{aligned}$$

- This is similar to what we did in electrostatics: $\vec{\nabla}\times\vec{E}(\vec{r}) = 0$ could be solved by writing

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

↑ electric potential.

(as $\vec{\nabla} \times \vec{\nabla} V(\vec{r}) = 0$).

In this case, $V(\vec{r})$ had a physical interpretation: $q V(\vec{r})$ = electric potential energy of charge q at $\vec{r}$

- The vector potential $\vec{A}(\vec{r})$ satisfying $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$ has no similar physical interpretation – it is just a mathematical convenience.

- We will come back to the vector potential $\vec{A}(\vec{r})$ when we consider EM waves

A consistency check on Ampere's law

In magnetostatics (constant currents, no "sources" or "sinks" of current),

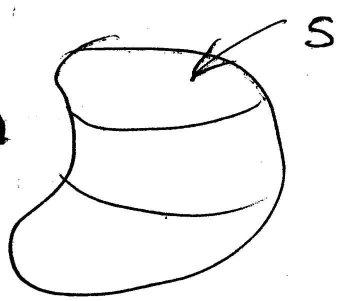
$$\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{\vec{j}(\vec{r})}{\varepsilon_0 c^2}$$ Ampere's ~~Gauss~~'s law

$j(\vec{r})$ = current density.

Since $\vec{\nabla} \cdot (\vec{\nabla} \times B(\vec{r})) = 0$ always,

$$\vec{\nabla} \cdot \vec{j}(\vec{r}) = 0.$$

Using Gauss's theorem, this implies the flux of $\vec{j}(\vec{r})$ through any closed surface is zero



$$\underbrace{\oint_S \vec{j}(\vec{r}) \cdot d\vec{S}}_{\text{flux through surface } S} = \int_V \underbrace{\vec{\nabla} \cdot \vec{j}(\vec{r})}_{0} \, d^3\vec{r} = 0$$

This makes sense: steady currents with no sources or sinks $\Rightarrow$ net current flowing into S = net current flowing out

- But in circuits, we have sources and sinks of currents (batteries, generators) ~~and sinks (resistors)~~ $\Rightarrow$ no longer magnetostatics. We ~~need~~ need to consider ELECTRODYNAMICS — which includes time varying charge densities $\rho(\vec{r}, t)$ + current densities $\vec{J}(\vec{r}, t) \Rightarrow$ time varying $\vec{E}$ and $\vec{B}$ fields, $\vec{E}(\vec{r}, t)$, $\vec{B}(\vec{r}, t)$.